α-chromatic symmetric functions

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Overview

There are two families of symmetric functions indexed by the set of Dyck paths:

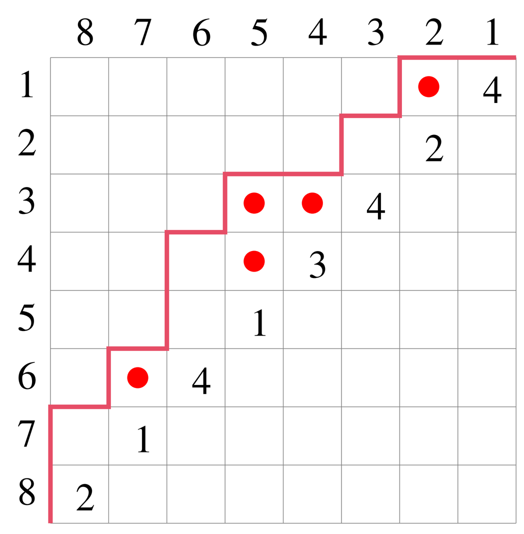
- the **chromatic quasisymmetric functions**

$$X_\pi(X;q) = \sum_{\substack{w\in\mathbb{Z}^n_{>0}\\ \text{proper}}} q^{\mathrm{inv}_\pi(w)} x_1^{w(1)} x_2^{w(2)} \cdots x_n^{w(n)},$$

- the **unicellular LLT polynomials**

$$\mathrm{LLT}_\pi(X;q) = \sum_{w\in\mathbb{Z}^n_{>0}} q^{\mathrm{inv}_\pi(w)} x_1^{w(1)} x_2^{w(2)} \cdots x_n^{w(n)},$$

where $\mathrm{inv}_\pi(w) = |\{(i,j) : (i,j)\in \mathrm{Area}(\pi),\ w(i)<w(j)\}|$.



$q^5 x_1^2 x_2^2 x_3 x_4^3$

Thereom. [Carlsson–Mellit, '2018]

$$X_\pi(X;q) = \frac{\mathrm{LLT}_\pi[(q-1)X;q]}{(q-1)^n}.$$

Definition. [α-chromatic symmetric functions]

$$X_\pi^{(\alpha)}(X;q) = \frac{\mathrm{LLT}_\pi[(q^\alpha-1)X;q]}{(q-1)^n} = X_\pi\left[\frac{q^\alpha-1}{q-1}X;q\right].$$

Motivation

- Haglund's conjecture:

$$\left\langle \frac{J_\mu(X;q,q^\alpha)}{(1-q)^n}, s_\lambda(X)\right\rangle \in \mathbb{N}[q]$$

- Haglund–Wilson conjecture: for any $\alpha\in\mathbb{N}$,

$$\left\langle \frac{J_\mu(X;q,q^{-\alpha})}{(q-1)^n}, s_\lambda(X)\right\rangle \in \mathbb{N}[q]$$

after some negative q-factors taken care of.

- Jaeseong's conjecture:

$$q^{\alpha\sum_i i\cdot\mu_i}\frac{J_\mu(X;q,q^{-\alpha})}{(q-1)^n} \text{ is } e\text{-positive.}$$

- Alexandersson–Haglund–Wang conjecture:

$$\left\langle \tilde{J_\mu}^{(\alpha)}(X), s_\lambda(X)\right\rangle = \sum_{k=0}^{n-1} a_k(\mu,\lambda)\binom{\alpha+k}{n} = \sum_{k=1}^{n} b_{n-k}(\mu,\lambda)\binom{\alpha}{k}k!,$$

with $a_k(\mu,\lambda),\ b_{n-k}(\mu,\lambda)\in\mathbb{N}$.

Parallel Universe?

- The **integral form** Macdonald polynomial $J_\mu(X;q,t)$ satisfies

$$J_\mu(X;q,t) = \sum_{\lambda\vdash n} K_{\lambda\mu}(q,t)\, s_\lambda[X(1-t)].$$

Then the **modified** Macdonald polynomial is defined by

$$H_\mu(X;q,t) = J_\mu\left[\frac{X}{1-t};q,t\right] = \sum_{\lambda\vdash n} K_{\lambda\mu}(q,t)\, s_\lambda(X).$$

- By the Carlsson–Mellit relation, we have

$$X_\pi(X;q) = \frac{\mathrm{LLT}_\pi[(q-1)X;q]}{(q-1)^n}.$$

e-positivity		Schur-positivity
$J_\mu\left[\frac{X}{1-t};q,t\right]$	$=$	$H_\mu(X;q,t)$
$(1-q)^n X_\pi\left[\frac{X}{q-1};q\right]$	$=$	$\mathrm{LLT}_\pi(X;q)$

Superization

Superization is a standard technique to deal with the *plethystic substitution.*

Proposition. If $G(X;q,t) = \sum_{\sigma\in S_n} c_\sigma F_{\mathrm{Des}(\sigma^{-1})}(X) \in \Lambda_n\otimes\mathbb{C}(q,t)$, then

1. $G(X;q,t) = \sum_{w\in\mathbb{Z}^n_+} c_{\mathrm{stan}(w)} X^w,$
2. $\langle G, h_\lambda(X)\rangle = G|_{m_\lambda} = \sum_{\substack{\sigma\in S_n\\ \sigma \text{ is a } \lambda\text{-shuffle}}} c_\sigma,$
3. $$\omega^Y G[X+Y;q,t] = \sum_{\sigma\in S_n} c_\sigma \tilde{F}_{\mathrm{Des}(\sigma^{-1})}(X,Y) = \sum_{\tilde{w}\in\mathscr{A}^n_\pm} c_{\mathrm{stan}(\tilde{w})} \prod_{\tilde{w}_i\in\mathscr{A}_+} x_{\tilde{w}_i} \prod_{\tilde{w}_i\in\mathscr{A}_-} y_{|\tilde{w}_i|},$$
4. $\langle G, e_\alpha h_\mu\rangle = \sum_{\substack{\sigma\in S_n\\ \sigma \text{ is a } (\mu,\alpha)\text{-shuffle}}} c_\sigma = \omega^Y G[X+Y;q,t]\big|_{X^\mu Y^\alpha}.$

Note that $\omega^Y G[X+Y;q] = G[X-\varepsilon Y;q]$.

Monomial expansion

Proposition. [HOY, '2025]

$$X_\pi\left[\frac{q^\alpha-1}{q-1};q\right] = q^{|\mathrm{area}(\pi)|}\prod_{i=1}^n[\alpha-a_i(\pi)]_q = q^{|\mathrm{area}(\pi)|}\prod_{i=1}^n[\alpha-b_i(\pi)]_q,$$

where $a_i(\pi)$ (resp. $b_i(\pi)$) is the number of cells below π and strictly above the diagonal in the i-th row (resp. i-th column).

From this, we obtain

$$X^{(\alpha)}(X;q) = \sum_{\lambda\vdash n}\sum_{\substack{\sigma\\ wt(\sigma)=\lambda}} q^{\mathrm{inv}_\pi(\sigma)}\prod_{i=1}^{\ell(\lambda)}\left(q^{|\mathrm{area}(\pi[i])|}\prod_{j=1}^{\lambda_i}[\alpha-\mathrm{area}(\pi[i])_j]_q\right)m_\lambda,$$

where $$\prod_{j=1}^n[\alpha-\mathrm{area}(\pi)_j]_q = q^{-|\mathrm{area}(\pi)|}\sum_{\substack{\sigma\in\{0,1,\dots,\alpha-1\}^n\\ \text{proper}}} q^{\mathrm{inv}_\pi(\sigma)+|\sigma|}.$$

Theorem. [HOY, '2025]

$$X_\pi^{(\alpha)}(X;q) = \sum_{\lambda\vdash n}\sum_{(w,\sigma)\in M(\lambda)\times S_n} q^{\mathrm{stat}_\pi(w,\sigma)}\begin{bmatrix}\alpha+\mathrm{asc}_\pi(w,\sigma)\\ n\end{bmatrix}_q m_\lambda(X)$$

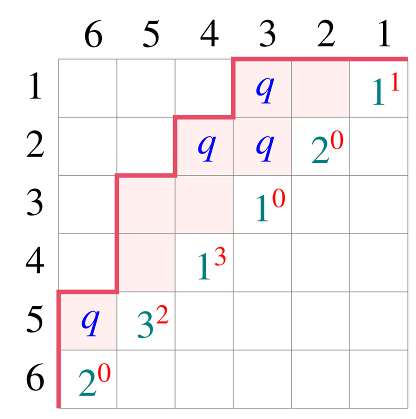
The bijection

Given $(c,d)\in\mathscr{C}^{(\alpha)}_{\pi,\lambda}$, the set of α-decorated proper colorings of weight λ, let σ_i be the index j such that (c_i,d_i) is the j-th element in the colexicographic order. Let $w=(w_1,\dots,w_n)$ be the word such that $w_i = c_{\sigma_i}$. We construct a bijection

$$\varphi : \mathscr{C}^{(\alpha)}_{\pi,\lambda} \to \cup_{k=0}^{n-1}\{(w,\sigma)\in M(\lambda)\times S_n : \mathrm{asc}(w,\sigma)=k\}\times\binom{[\alpha+k]}{n},$$

where $\binom{[\alpha+k]}{n} = \{\tau=(\tau_1,\dots,\tau_n)\,|\,0\le\tau_i\le\alpha+k-n,\ \tau_1\le\cdots\le\tau_n\}$, $\tau_i = \tilde{d_i}-(i-1)+\mathrm{asc}_\pi^{<i}(w,\sigma)$.

Example. For $\lambda=(3,2,1)$, $\alpha=3$:



$$\longrightarrow \begin{pmatrix}\dot{1} & \dot{2} & 2 & \dot{1} & 3 & 1\\ 3 & 2 & 6 & 1 & 5 & 4\end{pmatrix}\times(0,0,0,0,1,1)$$

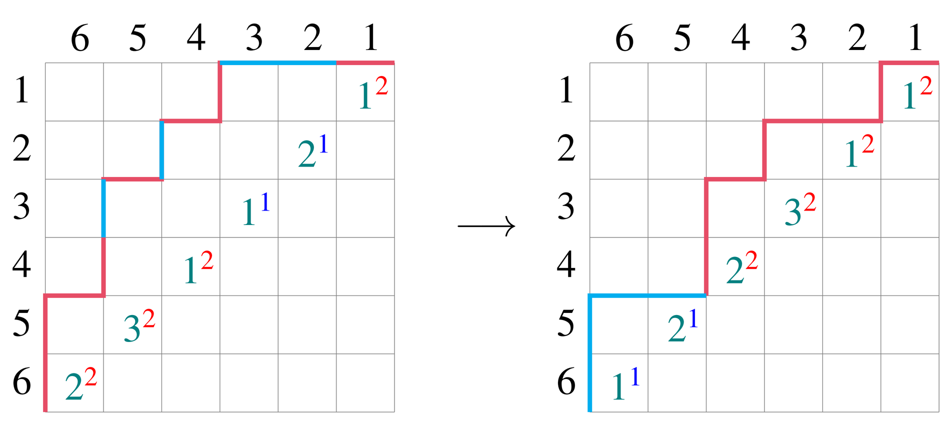
XY-technique

Given two sets of variables X and Y, let $XY=\{x_iy_j : i,\ j\ge 1\}$.

Theorem. [HOY, '2025]

$$X_\pi(XY;q) = \sum_{\lambda\vdash n} m_\lambda(X)\sum_{w\in M(\lambda)} q^{\mathrm{inv}_\pi(w)} X_{\beta(\pi,w)}(Y;q),$$

where the Dyck path $\beta(\pi,w)$ corresponds to the graph G_β obtained by removing all edges connecting vertices with different colors in G_π.



Corollary. [HOY, '2025] If we let

$$X_\pi(X;q) = \sum_{\lambda\vdash n} C_{\pi,\lambda}(\alpha)s_\lambda(X),$$

then for $\alpha\in\mathbb{N}$, $C_{\pi,\lambda}(\alpha)\in\mathbb{N}[q]$.

When $q=1$:

Theorem. [HOY, '2025]

$$X_\pi^{(\alpha)}(X;q=1) = \sum_{k=1}^n(\alpha)^{\underline{k}}\sum_{S\in S(n,k)} X_{\beta(\pi,S)}(X;1),$$

where $S(n,k)$ is the set of set-partitions of $[n]$ with k subsets and for $S=\{S^{(1)},\dots,S^{(k)}\}\in S(n,k)$, w_S is the word obtained by replacing elements in $S^{(i)}$ by i, then $\beta(\pi,S)=\beta(\pi,w_S)$.

Schur expansion

Proposition. [HOY, '2025] $s_\lambda\left[\frac{q^\alpha-1}{q-1}X\right]$ is Schur positive in the basis $\left\{\begin{bmatrix}\alpha+k\\ n\end{bmatrix}_q\right\}_{0\le k\le n-1}$.

Proof. Note that

$$s_\lambda[XY] = \sum_{\mu,\nu\vdash n} g^\lambda_{\mu,\nu}s_\mu[X]s_\nu[Y],$$

where $g^\lambda_{\mu,\nu}$ is the Kronecker coefficients which are known to be nonnegative. For $Y=\frac{q^\alpha-1}{q-1}$, the principal specialization of Schur functions has the following form

$$s_\nu\left[\frac{q^\alpha-1}{q-1}\right] = \sum_{T\in\mathrm{SYT}(\nu)} q^{\mathrm{maj}(T)}\begin{bmatrix}\alpha+n-1-d(T)\\ n\end{bmatrix}_q,$$

where $d(T)$ is the number of descents. □

Corollary. [HOY, '2025] If f is a Schur-positive symmetric function, then $f\left[\frac{q^\alpha-1}{q-1}X\right]$ is $\left\{\begin{bmatrix}\alpha+k\\ n\end{bmatrix}_q\right\}_{0\le k\le n-1}$-Schur positive and $\left\{\begin{bmatrix}\alpha\\ k\end{bmatrix}_q\right\}_{1\le k\le n}$-Schur positive.

In particular, $X_\pi^{(\alpha)}(X;q)$ is Schur positive in terms of aforementioned bases.

Conjecture. $X_\pi^{(\alpha)}(X;q)$ has a positive integral expansion in terms of $\{[\alpha]_q^{\underline{k}}s_\lambda\}_{1\le k\le n,\lambda\vdash n}$. Moreover, the Schur expansion of $X_\pi^{(\alpha)}(X;q)$ is of the form

$$X_\pi^{(\alpha)}(X;q) = \sum_{\lambda\vdash n}\left(\sum_{T\in\mathrm{SYT}(\lambda')} q^{\mathrm{stat}(T)}PR(B(T))\right)s_\lambda(X),$$

where $PR(B(T)) = \prod_{i=1}^n[\alpha+c_i-i+1]_q$ is the *rook product* of the Ferrers board $B(T)$ with the i-th column height c_i.

Application to rook theory

Rook Theory. Given a Ferrers board $B=(c_1,\dots,c_n)$, $0\le c_1\le\cdots\le c_n\le n$,

$$\prod_{i=1}^n[\alpha+c_i-i+1]_q = \sum_{k=1}^n r_{n-k}(B;q)[\alpha]_q^{\underline{k}} = \sum_{k=0}^{n-1}h_k(B;q)\begin{bmatrix}\alpha+k\\ n\end{bmatrix}_q,$$

where $r_k(B;q)$ is the k-th rook polynomial and $h_k(B;q)$ is the k-th hit polynomial.

A basic identity satisfied by the Gessel's fundamental quasisymmetric function F_D:

$$F_D\left[\frac{q^\alpha-1}{q-1}\right] = q^{n|D|-\sum_{i\in D}i}\begin{bmatrix}\alpha+n-1-|D|\\ n\end{bmatrix}_q.$$

Shareshian–Wachs found the fundamental expansion of $X_\pi(X;q)$:

$$X_\pi(X;q) = \sum_{\sigma\in S_n} q^{\mathrm{inv}_\pi(\sigma^{-1})}F_{\mathrm{PDes}^c_\pi(\sigma)}(X).$$

Hence we have

$$X_\pi\left[\frac{q^\alpha-1}{q-1};q\right] = \sum_{\sigma\in S_n} q^{\mathrm{stat}_{B_\pi}(\sigma)}\begin{bmatrix}\alpha+|\mathrm{PDes}_\pi(\sigma)|\\ n\end{bmatrix}_q,$$

where

$$\mathrm{stat}_{B_\pi}(\sigma) = \mathrm{inv}_\pi(\sigma^{-1})+n(n-1)-n|\mathrm{PDes}_\pi(\sigma)| - \sum_{i:\sigma_i\notin\mathrm{PDes}_\pi(\sigma)} i.$$

Since we proved

$$X_\pi\left[\frac{q^\alpha-1}{q-1};q\right] = q^{|\mathrm{area}(\pi)|}\prod_{i=1}^n[\alpha-a_i(\pi)]_q = q^{|\mathrm{area}(\pi)|}\prod_{i=1}^n[\alpha-b_i(\pi)]_q,$$

$\mathrm{stat}_{B_\pi}(\sigma)$ gives a combinatorial interpretation for the q-hit polynomial in the *product formula.*

[HOY, '2025] α-Chromatic Symmetric Functions. *Int. Math. Res. Not. IMRN*, (11):rnaf144-, 2025.