The Isomorphism Problem for (Co)adjoint Schubert Varieties

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Objectives

We classify isomorphism classes of Schubert varieties coming from adjoint and coadjoint partial flag varieties across all Dynkin types via Hasse diagrams given by the Chevalley formula.

Backgrounds

Let G be a reductive group; fix a Borel subgroup B and a maximal torus T. The triple (G, B, T) determines a root system Φ with root basis Δ and Weyl group W. A subset $I \subseteq \Delta$ gives rise to subgroups $W_I \subseteq W$ and $P \subseteq G$ (parabolic subgroup).

The quotient space G/P is called a **partial flag variety**, and the closure of orbits $X_w = \overline{BwP/P}$ are called **Schubert varieties**. The Schubert varieties in G/P are indexed by W/W_I, or equivalently by W^I, the set of all minimal length coset representatives of W/W_I.

Let Θ be the highest (long) root in the root system. The weight $\varpi = \Theta$ is called **adjoint weight**. The partial flag variety G/P is called **adjoint** if P is the parabolic subgroup associated to ϖ.

The **coadjoint** case is almost the same, except that 'long' shall be replaced with 'short'; denote by θ the highest short root.

Key Properties

In the adjoint case, $W/W_I = W/\operatorname{Stab}(\Theta)$ acts on Φ_{long}, which gives a bijection

$$W/W_I \text{ (or } W^I) \longrightarrow \Phi_{\text{long}}$$
$$w \longmapsto w(-\Theta).$$

A similar property holds for the coadjoint case, with Θ replaced by θ, 'long' replaced by 'short'.

This allows us to index the Schubert varieties by long(short) roots: if $\alpha = w(-\Theta)$(or $w(-\theta)$), then define $X_\alpha := X_w$.

Chevalley Formula

For a Schubert variety X_w, an integral basis of $\mathrm{CH}_*(X_w)$ is indexed by the Schubert classes $[X_u]$ such that $X_u \subseteq X_w$; there is a Chevalley formula telling us how $\operatorname{Pic}(X_w)$ acts on $\mathrm{CH}_*(X_w)$ ([1],[2]).

In adjoint(resp. coadjoint) cases, with a carefully chosen divisor $D' \in \operatorname{Pic}(X_\beta)$, we get a simplified Chevalley formula: for $\alpha \in \Phi_{\text{long(resp. short)}}$ we have

$$D' \cdot [X_\alpha] = \begin{cases} \displaystyle\sum_{\gamma \in \Delta,\ (\gamma^\vee, \alpha) > 0} (\gamma^\vee, \alpha)[X_{s_\gamma \alpha}] & \text{if } \alpha \notin \Delta_{\text{long(resp. short)}}; \\ \displaystyle\sum_{\gamma \in \Delta_{\text{long (resp. short)}}} (\gamma^\vee, \alpha)[X_{-\gamma}] & \text{if } \alpha \in \Delta_{\text{long(resp. short)}}. \end{cases}$$

Let D be D' multiplied by some coefficient; it can be shown that D only depends on the algebraic structure of X_β.

Chevalley-Hasse Diagrams

For an adjoint Schubert variety X_β, we then construct the Chevalley-Hasse diagram P_β as follows:

Vertices: all Schubert classes $[X_\alpha]$ such that $X_\alpha \subseteq X_\beta$. These classes are indexed by those long roots α such that $\alpha \leq \beta$ when they are of the same sign, or such that $\operatorname{Supp}(\alpha) \cup \operatorname{Supp}(\beta)$ is connected when they are of different signs. The vertices admit a natural partial order of inclusion relationship.

Edges: for two vertices corresponding to the long roots α, α' in the diagram, we draw n oriented edge(s) from α to α' if the coefficient of $[X_\alpha]$ in $D \cdot [X_{\alpha'}]$ is n.

The Chevalley formula allows us to compute the diagrams in combinatorial ways. Indeed, we only need to compute the diagram P for $X = X_\Theta$, and P_β will be the full subgraph of P topped at β.

A similar construction is available for coadjoint Schubert varieties, with 'long' replaced by 'short'.

Main Theorem

Let $X_\alpha \subseteq X$ and $Y_\beta \subseteq Y$ be Schubert varieties coming from adjoint or coadjoint partial flag varieties. Then

$$X_\alpha \simeq Y_\beta \text{ if and only if } P(X_\alpha) \simeq P(Y_\beta).$$

Examples

We subscript the simple roots in each Dynkin type in a standard order following ([3]).

Figure 1 shows algebraic isomorphisms across some Schubert varieties of type B, C, D. In particular, the theorem is applicable for comparison across **adjoint and coadjoint** types.

Figure 2,3,4, on the other hand, demonstrate three pairwise non-isomorphic Schubert varieties although their Chow groups are free of the same rank; the point is that there are no isomorphisms of the Chow groups preserving the action of D.

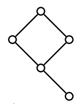

Figure 1. $P(B_4, -\alpha_2 - \alpha_3 - 2\alpha_4) \simeq P(C_4, -\alpha_2 - \alpha_3 - \alpha_4) \simeq P(D_4, -\alpha_2 - \alpha_4)$

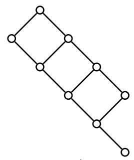

Figure 2. $P(C_4, -\alpha_2)$

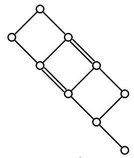

Figure 3. $P(B_4, -\alpha_2)$

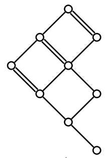

Figure 4. $P(B_4, -\alpha_3)$

In **Figure 5,6** we examine Chevalley-Hasse diagrams of the partial flag varieties $X(F_4, \theta)$ and $X(E_7, \theta)$. While the diagrams themselves are highly non-isomorphic, the patterns in the coloured parts indicate that there are isomorphisms between Schubert varieties indexed by some negative roots.

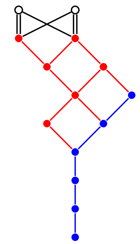

Figure 5. $P(F_4, \theta)$, lower half

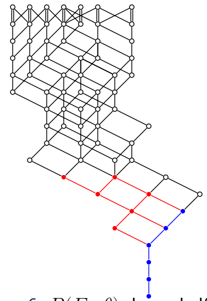

Figure 6. $P(E_7, \theta)$, lower half

Proving Strategies

'Only if' part: since the selected basis $\{X_\alpha\}$, the intersection product and the chosen divisor D all depend only on the algebraic structure of X_β, so does the Chevalley-Hasse diagram P_β.

'If' part: two techniques, namely **minimal embeddings** and **foldings**, are applied to embed Schubert varieties with the same Chevalley-Hasse diagrams into a same (possibly non-(co)adjoint) Schubert variety. For example, in **figure 5,6**, minimal embeddings identify Schubert varieties corresponding to the blue vertices and foldings deal with the red ones. A type-by-type analysis is needed.

Problem

Is there a type-independent proof for generalizations?

References

[1] W. Fulton, R. MacPherson, F. Sottile, and B. Sturmfels. Intersection theory on spherical varieties. J. Algebraic Geom., 4(1):181–193, 1995.

[2] W. Fulton and C. Woodward. On the quantum product of schubert classes. J. Algebraic Geom., 13(4):641–661, 2004.

[3] Pieter Belmans. Grassmannian.info — a periodic table of (generalised) grassmannians, 2025.

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