Rank-selected Segre powers of the Boolean lattice

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Segre powers preserve CM

Let P be a **bounded** partially ordered set , with rank function ρ. For each $t \geq 1$, **define** the t-fold **Segre power** of P, $P^{(t)}$, to be the induced subposet of the t-fold product poset $P \times \cdots \times P$ consisting of the elements

$$\{(x_1, \ldots, x_t) : \rho(x_1) = \cdots = \rho(x_t)\} = \underbrace{P \circ \cdots \circ P}_{t}.$$

$P^{(t)}$ is bounded and ranked with the rank function ρ.

$$(x_1, \ldots, x_t) < (y_1, \ldots, y_t) \iff x_i < y_i, 1 \leq i \leq t$$

Definition (K. Baclawski 1980, R. Stanley 1977): Let P be as above, and let $\Bbbk$ be any field. Then P is **Cohen-Macaulay** over $\Bbbk$ if for every open interval (x, y), the reduced simplicial homology $\tilde{H}_i(x, y)$ of (x, y) **vanishes in all but the top dimension** $\mathrm{rk}(y) - \mathrm{rk}(x) - 2$.

Theorem (A. Björner & V. Welker 2005): If P is (homotopy) Cohen-Macaulay over $\Bbbk$, then so is $P \circ P$.

P Cohen-Macaulay $\Rightarrow$ there is ONE homology module, on which the automorphism group of P, $\mathrm{Aut}(P)$, acts.

The action of $\mathrm{Aut}(P)$

If $G = \mathrm{Aut}(P)$, then $G^{\times t}$ acts on $P^{(t)}$, and G itself acts diagonally on $P^{(t)}$. Let $P = B_n$, the Boolean lattice of subsets of $[n]$. Then $\mathrm{Aut}(P) = \mathfrak{S}_n$, the symmetric group.

GOAL: To determine the actions of $\mathfrak{S}_n^{\times t}$ and of the diagonal subgroup $\mathfrak{S}_n$ on the **unique nonvanishing homology** of the t-fold Segre power of B_n.

FACT: $\mathfrak{S}_n$ acts on the unique nonvanishing homology of B_n like the sign representation.

The top homology of $B_n^{(t)}$: dimension

Definition: An **ascent** of a permutation $\sigma \in \mathfrak{S}_n$ is an i, $1 \leq i \leq n-1$, with $\sigma(i) < \sigma(i+1)$.
For $t \geq 1$, define $w_n^{(t)}$ to be the number of t-tuples of permutations in $\mathfrak{S}_n$ with **no common ascent**.

Note: $w_n^{(1)} = 1$. In 1976, Stanley showed the following:

Proposition: The Möbius number of $B_n^{(t)}$ is $(-1)^{n-2} w_n^{(t)}$. These numbers satisfy the generating function

$$\sum_{n \geq 0} w_n^{(t)} \frac{z^n}{n!^t} = \frac{1}{f(z)}, \quad \text{where } f(z) = \sum_{n \geq 0} (-1)^n \frac{z^n}{n!^t}.$$

The numbers $w_n^{(t)}$ were studied by Abramson and Promislow (1978). The case $t = 2$ was studied (more famously) by Carlitz, Scoville and Vaughan (1976).
The dimensions table (Compiled from `oeis` A212855).

$t \backslash n$	0	1	2	3	4	5
$t=1$	1	1	1	1	1	1
2	1	1	3	19	211	3651
3	1	1	7	163	8983	966751
4	1	1	15	1135	271375	

Tab. 1: The numbers $w_n^{(t)}$ for $0 \leq n \leq 5$, $1 \leq t \leq 4$

The product Frobenius characteristic

- R^n: the v.s. spanned by the class functions of $\mathfrak{S}_n$ over $\mathbb{Q}$.
- $\underline{\mathbf{n}} = (n_1, \ldots, n_t)$ in $\mathbb{Z}^t_{\geq 0}$, $R^{\underline{\mathbf{n}}} := \bigotimes_{i=1}^t R^{n_i}$, $\mathfrak{S}_{\underline{\mathbf{n}}} = \times_{i=1}^t \mathfrak{S}_{n_i}$.
- (X^i), $i = 1, \ldots, t$: t sets of variables.
- $\Lambda^{n_i}(X^i)$: ring of symmetric functions in the ith set of variables (X^i), of homogeneous degree n_i.

Identify: $\bigotimes_{i=1}^t f_{n_i}(X^i) \mapsto \prod_{i=1}^t f_{n_i}(X^i)$.

Definition (Li-Sundaram 2024): Let $f_{n_i} \in R^{n_i}$ and define the map $\mathrm{Pch} : R^{\underline{\mathbf{n}}} = \bigotimes_{i=1}^t R^{n_i} \to \bigotimes_{i=1}^t \Lambda^{n_i}(X^i)$ as:

$$\mathrm{Pch}\left(\bigotimes_{i=1}^t f_{n_i}\right) := \prod_{i=1}^t \mathrm{ch}(f_{n_i})(X^i),$$

- ch is the ordinary Frobenius characteristic map on R^{n_i}.

For the $(\times_{i=1}^t \mathfrak{S}_{n_i})$-irreducible $\bigotimes_{i=1}^t \chi^{\lambda^i}$ indexed by the t-tuple $\underline{\lambda} = (\lambda^1, \ldots, \lambda^t)$:

$$\mathrm{Pch}\left(\bigotimes_{i=1}^t \chi^{\lambda^i}\right) = \prod_{i=1}^t s_{\lambda^i}(X^i)$$

Product characteristic: example

Example: Let $t = 2$, the **regular representation** ψ of $\mathfrak{S}_2 \times \mathfrak{S}_3$ decomposes into irreducibles as $\chi^{((2),(3))} + \chi^{((1^2),(3))} + 2\chi^{((2),(2,1))} + 2\chi^{((1^2),(2,1))} + \chi^{((2),(1^3))} + \chi^{((1^2),(1^3))}$. We have

$$\begin{aligned}\mathrm{Pch}(\psi) &= s_{(2)}(X^1)s_{(3)}(X^2) + s_{(1^2)}(X^1)s_{(3)}(X^2) \\ &+ 2s_{(2)}(X^1)s_{(2,1)}(X^2) + 2s_{(1^2)}(X^1)s_{(2,1)}(X^2) \\ &+ s_{(2)}(X^1)s_{(1^3)}(X^2) + s_{(1^2)}(X^1)s_{(1^3)}(X^2) \\ &= h_1^2(X^1)h_1^3(X^2).\end{aligned}$$

The top homology of $B_n^{(t)}$

Fix $t \geq 1$. The direct product $\mathfrak{S}_n^{\times t}$ acts on the top homology $\tilde{H}_{n-2}(B_n^{(t)})$ of the t-fold Segre power $B_n^{(t)}$. Let $\beta_n^{(t)}$ be its product Frobenius characteristic.

Theorem (Li-Sundaram 2024): Set $\beta_0^{(t)} = 1$. Then $\beta_n^{(t)}$ satisfies the recurrence

$$\sum_{i=0}^n (-1)^i \beta_i^{(t)} \prod_{j=1}^t h_{n-i}(X^j) = 0.$$

A new homomorphism

Define, for each $t \geq 1$,
- $Z_n^{(t)} := \prod_{j=1}^t h_n(X^j)$.
- For each $\lambda \vdash n$, $Z_\lambda^{(t)} = \prod_j Z_{\lambda_j}^{(t)} = \prod_{j=1}^t h_\lambda(X^j)$.
- A map $\Phi_t : \Lambda(X) \to \bigotimes_{j=1}^t \Lambda(X^j)$ by multiplicatively and linearly extending

$$\Phi_t(h_n) := \prod_{j=1}^t h_n(X^j) = Z_n^{(t)}.$$

Proposition (Li-Sundaram 2024): Fix $t \geq 1$. The map $\Phi_t : \Lambda(X) \to \bigotimes_{j=1}^t \Lambda(X^j)$ is an injective ring homomorphism. It satisfies $\Phi_t(h_\lambda) = Z_\lambda^{(t)}$ and $\Phi_t(e_n) = \beta_n^{(t)}$. Hence, $\{\beta_n^{(t)}\}_n$ is an algebraically independent set.

Note: $\Phi_t(s_\lambda)$ does not always expand positively into irreducibles of $\mathfrak{S}_n^{\times t}$. If $\lambda = 322$, then the multiplicity of $(43, 61)$ in the module $\Phi_2(s_{322})$ is -1.

Irreducible decomposition of $\beta_n^{(t)}$

For $\lambda \vdash n$ with $m_i(\lambda)$ parts of size i and number of parts $\ell(\lambda)$, define the integer

$$c_\lambda = (-1)^{n-\ell(\lambda)} \frac{\ell(\lambda)!}{\prod_i m_i(\lambda)!} = (-1)^{n-\ell(\lambda)} \binom{\ell(\lambda)}{m_1(\lambda), m_2(\lambda, \ldots}.$$

Theorem: (Li-Sundaram 2024): Let $\beta_n^{(t)}$ be the product Frobenius characteristic of the top homology of $B_n^{(t)}$. Then
(1) $\beta_n^{(t)} = \sum_{\lambda \vdash n} c_\lambda Z_\lambda^{(t)}$.
(2) The multiplicity of the $\mathfrak{S}_n^{\times t}$-irreducible indexed by the t-tuple of partitions $\underline{\mu} = (\mu^1, \ldots, \mu^t)$, $\mu^j \vdash n$, $1 \leq j \leq t$,
- in $\tilde{H}_{n-2}(B_n^{(t)})$ equals $c_{\underline{\mu}}^t = \sum_{\lambda \vdash n} c_\lambda \prod_{j=1}^t K_{\mu^j, \lambda}$, where $K_{\mu,\nu}$ is the Kostka number.
- in the (possibly virtual) module with product Frobenius characteristic $\Phi_t(s_\lambda)$ is $\sum_{\nu \vdash n} \mathcal{M}(s,h)_{\lambda,\nu} \prod_{j=1}^t K_{\mu^j,\nu}$, where $\mathcal{M}(s,h)$ is the transition matrix from Schur functions to homogeneous functions.

Example: $\beta_3^{(2)}$ and $\Phi_t(s_{321})$

Recompute $\beta_3^{(2)}$, using $e_3 = h_3 - 2h_2h_1 + h_1^3$. Note that $h_2h_1 = s_{(3)} + s_{(2,1)}$ and $h_1^3 = s_{(3)} + 2s_{(2,1)} + s_{(1^3)}$.

$$\begin{aligned}\beta_3^{(2)} &= \Phi_t(e_3) = Z_{(3)}^{(2)} - 2Z_{(2,1)}^{(2)} + Z_{(1^3)}^{(2)} \\ &= s_{(3)}(X^1)s_{(3)}(X^2) \\ &- 2(s_{(3)}(X^1) + s_{(2,1)}(X^1))(s_{(3)}(X^2) + s_{(2,1)}(X^2)) \\ &+ \left(s_{(3)}(X^1) + 2s_{(2,1)}(X^1) + s_{(1^3)}(X^1)\right)\left(s_{(3)}(X^2) \right.\\ &\left. + 2s_{(2,1)}(X^2) + s_{(1^3)}(X^2)\right) \\ &= h_3(X^1)e_3(X^2) + e_3(X^1)h_3(X^2) + e_3(X^1)e_3(X^2) + s_{(2,1)}(X^1)s_{(2,1)}(X^2) \\ &+ 2\, s_{(2,1)}(X^1)e_3(X^2) + 2\, e_3(X^1)s_{(2,1)}(X^2)\end{aligned}$$

Let $\lambda = (3, 2, 1)$. The multiplicity of the t-tuple of partitions $(\mu^1, \ldots, \mu^t)$ of 6 in the module with product Frobenius characteristic $\Phi_t(s_\lambda)$, is

$$\prod_{j=1}^t K_{\mu^j,321} - \prod_{j=1}^t K_{\mu^j,33} - \prod_{j=1}^t K_{\mu^j,411} + \prod_{j=1}^t K_{\mu^j,51},$$

since $s_{(3,2,1)} = h_{321} - h_{33} - h_{411} + h_{51}$.

The action of $\mathfrak{S}_n$ on $\tilde{H}_{n-2}(B_n^{(t)})$

Let $g_{\underline{\mu}}^\lambda$ denote the Kronecker coefficient $\langle \chi^\lambda, \prod_{j=1}^t \chi^{\mu_j} \rangle$, for a t-tuple $(\mu^1, \ldots, \mu^t)$ of partitions of n, $\lambda \vdash n$, and irreducible characters χ^ν of $\mathfrak{S}_n$.
Let $*$ denote the internal product in the ring of symmetric functions $\Lambda^n(X)$ in a single set of variables X.

Theorem (Li-Sundaram 2024): For the diagonal $\mathfrak{S}_n$-action on $\tilde{H}_{n-2}(B_n^{(t)})$,

$$\mathrm{ch}\,\tilde{H}_{n-2}(B_n^{(t)}) = \sum_{\underline{\mu}} c_{\underline{\mu}}^t g_{\underline{\mu}}^\lambda s_\lambda = \sum_{\underline{\mu}} c_{\underline{\mu}}^t s_{\mu^1} * \cdots * s_{\mu^t}.$$

Sum is over all $\underline{\mu} = (\mu^1, \ldots, \mu^t)$, $\mu^j \vdash n$, $1 \leq j \leq t$.

Data on diagonal action

- For $n = 2$: $\tilde{H}_0(B_2^{(t)}) = 2^{t-1}\chi^{(1^2)} + (2^{t-1} - 1)\chi^{(2)}$, in agreement with the case $t = 1$. In fact it is e-positive:

$$\mathrm{ch}\,\tilde{H}_0(B_2^{(t)}) = (2^{t-1} - 1)e_{(1,1)} + e_2.$$

The dimension is $w_2^{(t)} = 2^t - 1$. (rank 2 poset...)

- For $n = 3$: $\tilde{H}_1(B_3^{(t)}) = 2^{t-1}\chi^{(1^3)} + (2^{t-1} - 1)\chi^{(3)} + (2^t - 2)\sum_{r=1}^t \binom{t}{r} (\chi^{(2,1)})^{\otimes r}$, again agreeing with the case $t = 1$, and it is e-positive:

$$\mathrm{ch}\,\tilde{H}_1(B_3^{(t)}) = (6^{t-1} - 3^{t-1})e_{(1,1,1)} + e_3.$$

The dimension is $w_3^{(t)} = 6(6^{t-1} - 3^{t-1}) + 1$.
See oeis A248225, A127222 for $\{6^t - 3^t\}$

- For $n \geq 4$, SageMath shows the e-positivity breaks down, even for $t = 2$:
$\tilde{H}_2(B_4^{(2)}) = 10\chi^{(1^4)} + 9\chi^{(4)} + 26\chi^{(3,1)} + 18\chi^{(2,2)} + 26\chi^{(2,1^2)}$, and $\mathrm{ch}\,\tilde{H}_2(B_4^{(2)}) = 9e_{(1,1,1,1)} - \mathbf{e}_{(\mathbf{2,1,1})} + e_{(2,2)} + e_4$.

Rank-selection

Let P_J be the **rank-selected subposet** of P consisting of elements in the rank-set J, together with $\hat{0}$ and $\hat{1}$.

FACTS:
- If P is Cohen-Macaulay, so is P_J for every subset J.
- (Stanley) The module of maximal chains $\alpha_P(J)$ and the top homology module $\beta_P(J)$ of $P(J)$ are related:

$$\alpha_P(J) = \sum_{U \subseteq J} \beta_P(U), \quad \beta_P(J) = \sum_{U \subseteq J} (-1)^{|J|-|U|} \alpha_P(U).$$

For the poset $P = B_n^{(t)}$ we have:

Theorem (Li-Sundaram 2024):
$\alpha_n^{(t)}(J) = \sum_{\lambda \vdash n} \Phi_t(s_\lambda) \, |\{SYT\ \tau \text{ of shape } \lambda : \mathrm{Des}(\tau) \subseteq J\}|$;
$\beta_n^{(t)}(J) = \sum_{\lambda \vdash n} \Phi_t(s_\lambda) \, |\{SYT\ \tau \text{ of shape } \lambda : \mathrm{Des}(\tau) = J\}|$.

Theorem (Li-Sundaram 2024): The product Frobenius characteristic Pch $\tilde{H}(B_n^{(t)}(J))$ satisfies the following recurrence.

$$\beta_n^{(t)}(J) + \beta_n^{(t)}(J \setminus \{j_r\}) = \beta_{j_r}^{(t)}(J \setminus \{j_r\}) \prod_{i=1}^t h_{n-j_r}(X^i).$$

Stable principal specialisation

The **stable principal specialisation** ps f of a symmetric function f in variables $x_1, x_2 \ldots$ is $f(1, q, q^2, \ldots)$.
Let $B_{n,q}$ denote the lattice of subspaces of an n-dimensional vector space over a field with q elements.

Theorem (Li-Sundaram 2024): For the rank-selected homology module $\beta_n^{(t)}(J)$, ps $\beta_n^{(t)}(J)$ and the rank-selected Betti number $\tilde{\beta}_{B_{n,q}^{(t)}}(J)$ are related by

$$\mathrm{ps}\,\beta_n^{(t)}(J) = \frac{\tilde{\beta}_{B_{n,q}^{(t)}}(J)}{\prod_{i=1}^n (1-q^i)^t}.$$