q-deformation of graphic arrangements

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Motivation

It is classically known that there are similarities between subsets of $[n] = \{1, \dots, n\}$ and linear subspaces of $\mathbb{F}_q^n$, which is sometimes called the "q-analogue". We start with pointing out further similarities between chromatic polynomials for graphs and characteristic polynomials for hyperplane arrangements over finite fields. A typical example of an arrangement is the **braid arrangement** $\mathcal{B}_\ell$ in $\mathbb{R}^\ell$ whose defining polynomial is the **Vandermonde determinant**, i.e.,

$$Q(\mathcal{B}_\ell) = \prod_{1 \le i < j \le \ell} (x_j - x_i) = \begin{vmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{\ell-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{\ell-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_\ell & x_\ell^2 & \dots & x_\ell^{\ell-1} \end{vmatrix}.$$

The characteristic polynomial of the braid arrangement $\mathcal{B}_\ell$ is

$$\chi(\mathcal{B}_\ell, t) = t(t-1)(t-2)\cdots(t-\ell+1).$$

There are mysterious similarities between the braid arrangements and the arrangements consisting of all hyperplanes in vector spaces over finite fields. Let q be a prime power and $\mathbb{F}_q$ the finite field of order q. Define the arrangement $\mathcal{A}_{\text{all}}(\mathbb{F}_q^\ell)$ as the set of all hyperplanes in $\mathbb{F}_q^\ell$. Its defining polynomial is the determinant of the **Moore matrix**, i.e.,

$$Q\left(\mathcal{A}_{\text{all}}(\mathbb{F}_q^\ell)\right) = \prod_{i=1}^{\ell} \prod_{c_1, \dots, c_{i-1} \in \mathbb{F}_q} (c_1 x_1 + \dots + c_{i-1} x_{i-1} + x_i)$$

$$= \begin{vmatrix} x_1 & x_1^q & x_1^{q^2} & \dots & x_1^{q^{\ell-1}} \\ x_2 & x_2^q & x_2^{q^2} & \dots & x_2^{q^{\ell-1}} \\ \vdots & \vdots & \vdots & & \vdots \\ x_\ell & x_\ell^q & x_\ell^{q^2} & \dots & x_\ell^{q^{\ell-1}} \end{vmatrix}.$$

The characteristic polynomial of $\mathcal{A}_{\text{all}}(\mathbb{F}_q^\ell)$ is

$$\chi\left(\mathcal{A}_{\text{all}}(\mathbb{F}_q^\ell), t\right) = (t-1)(t-q)(t-q^2)\cdots(t-q^{\ell-1}).$$

By formally replacing q^k in the expressions of $Q\left(\mathcal{A}_{\text{all}}(\mathbb{F}_q^\ell)\right)$ and $\chi\left(\mathcal{A}_{\text{all}}(\mathbb{F}_q^\ell), t\right)$ with k, we obtain the expressions for $Q(\mathcal{B}_\ell)$ and $\chi(\mathcal{B}_\ell, t)$.
Note that $\chi(\mathcal{B}_\ell, \ell) = \ell! = |\mathfrak{S}_\ell|$ and $\chi\left(\mathcal{A}_{\text{all}}(\mathbb{F}_q^\ell), q^\ell\right) = (q^\ell - 1)(q^\ell - q)\cdots(q^\ell - q^{\ell-1}) = |GL_\ell(\mathbb{F}_q)|$. It is worth mentioning that the permutation group $\mathfrak{S}_\ell$ is considered as the "$\mathbb{F}_1$-version" of the general linear group $GL_\ell(\mathbb{F}_q)$ [5]. For more details in the theory of hyperplane arrangements, see[2].

Proposition 1-1

Suppose $\chi(G, k) = 0$ for some $k \in \mathbb{Z}_{\ge 0}$. Then $\chi(G, j) = 0$ for $0 \le j \le k$.

There is a q-version of Proposition 1-1.

Proposition 1-2

Let $\mathcal{A}$ be an arrangement in $\mathbb{F}_q^\ell$. If $\chi(\mathcal{A}, q^k) = 0$ for some $k \in \mathbb{Z}_{\ge 0}$, then $\chi(\mathcal{A}, q^j) = 0$ for any $0 \le j \le k$.[6]

Proposition 2-1

Let $t^{\underline{i}}$ be the falling factorial such that for each $i \in \mathbb{Z}_{>0}$, $t^{\underline{i}} \coloneqq t(t-1)\cdots(t-i+1)$. Suppose $\chi(G, t) = \sum_{i=1}^{\ell} c_i t^{\underline{i}}$. Then c_i coincides with the number of stable partitions of G into i blocks, where a stable partition of G is a set partition of the vertex set such that no edge connects vertices within the same block[3].
In other words, c_i coincides with the number of i-dimensional subspaces in $L(\mathcal{B}_\ell)$ that are not contained in any hyperplanes in $\mathcal{A}_G$.

Proposition 2-2

Let $\mathcal{A}$ be an arrangement in $\mathbb{F}_q^\ell$ and $t_q^{\underline{i}} \coloneqq \chi(\mathcal{A}_{\text{all}}(\mathbb{F}_q^i), t) = (t-1)(t-q)\cdots(t-q^{i-1})$. Suppose $\chi(\mathcal{A}, t) = \sum_{i=0}^{\ell} c_i t_q^{\underline{i}}$. Then c_i is the number of i-dimensional subspaces in $\mathbb{F}_q^\ell$ that are not contained in any hyperplanes in $\mathcal{A}$.

Definitions

Definition 1

*Define the **graphical arrangement** $\mathcal{A}_G$ in $\mathbb{R}^\ell$ by*

$$\mathcal{A}_G \coloneqq \{\{x_i - x_j = 0\} | \{i, j\} \in E_G\}.$$

Note that every subarrangement of the braid arrangement $\mathcal{B}_\ell$ is of the form $\mathcal{A}_G$ and it is well known that the chromatic polynomial $\chi(G, t)$ coincides with the characteristic polynomial $\chi(\mathcal{A}_G, t)$.

Definition 2

*We can define a **q-deformation of graphical arrangement** $\mathcal{A}_G^q$ in $\mathbb{F}_q^\ell$ as follows.*

$$\mathcal{A}_G^q \coloneqq \bigcup_{\{i_1, \dots, i_r\}} \{\{a_{i_1} x_{i_1} + \dots + a_{i_r} x_{i_r} = 0\} | a_{ij} \in \mathbb{F}_q^\times, j = 1, 2, \cdots, r\},$$

where $\{i_1, \dots, i_r\}$ runs over all cliques of G.

Conjecture and Main Theorem

Conjecture 1

The characteristic polynomial of the q-deformation $\chi(\mathcal{A}_G^q, t)$ is a polynomial in q and t, such that

$$\lim_{q \to 1} \frac{\chi(\mathcal{A}_G^q, q^t)}{(q-1)^\ell} = \chi(G, t).$$

The proof of the conjecture is still unclear but we proved a weaker version.

Main Theorem

For any $k \in \mathbb{Z}_{\ge 0}$, and prime power q,

$$\frac{\chi(\mathcal{A}_G^q, q^k)}{(q-1)^\ell} \equiv \chi(G, k) \pmod{q-1}.$$

Results Supporting the Conjecture

Graphs can be seen as simplicial complexes with faces made by cliques. Denote this **clique complex** by Δ_G. Note that all simplicial complexes are subcomplexes of some clique complex.

Definition 3

For a simplicial complex Δ, any face $\mathcal{F} \in \Delta$ can be seen as the clique complex of some graph, say $G_\mathcal{F}$. The q-deformation of Δ is defined as

$$\mathcal{A}_\Delta^q \coloneqq \bigcup_{\mathcal{F} \in \Delta} \mathcal{A}_{G_\mathcal{F}}^q.$$

Note that the q-deformation of the clique complex of graph G is the same as the q-deformation of $\mathcal{A}_G$. The conjecture can also be extended to the q-deformation of any simplicial complex.

Proposition 3

Let $G = ([\ell], E)$ be a graph, Δ_G^1 be the 1-skeleton of the clique complex Δ_G, i.e., the subcomplex $E \cup [\ell] \subseteq \Delta_G$. Then for any edge $e \in E_G$), we have

$$\chi(\mathcal{A}_{\Delta_G^1}^q, t) = \chi(\mathcal{A}_{\Delta_{G \setminus e}^1}^q, t) - (q-1)\chi(\mathcal{A}_{\Delta_{G/e}^1}^q, t).$$

Hence by computation, Δ_G^1 satisfies the conjecture for any graph G.

Corollary 1

Let G be a triangle-free graph, easy to see $\mathcal{A}_G^q = \mathcal{A}_{\Delta_G^1}^q$, hence any triangle-free graph satisfies the conjecture.

Proposition 4

Let G be a chordal graph, then we can write $\chi(G, t) = \chi(\mathcal{A}_G, t) = (t - e_1)(t - e_2)\cdots(t - e_\ell)$. In this case,

$$\chi(\mathcal{A}_G^q, t) = (t - q^{e_1})(t - q^{e_2})\cdots(t - q^{e_\ell}).$$

Hence satisfies the conjecture. The proof is by giving the free basis of the logarithmic vector field of the graphic arrangements[4] and the corresponding q-deformations.

Proposition 5

Let G be a graph on $[\ell]$ and $G + K_m$ be the join of graph G and the complete graph K_m. Then

$$\chi(\mathcal{A}_{G+K_m}^q, t) = (t-1)(t-q)\cdots(t-q^{m-1})q^{m\ell}\chi(\mathcal{A}_G^q, q^{-m}t).$$

While for the chromatic polynomial, we have

$$\chi(G + K_m, t) = t(t-1)\cdots(t-m+1)\chi(G, t-m).$$

Hence if the graph G satisfies the conjecture, the join of G with complete graphs also satisfies the conjecture.

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