A new class of magic positive Ehrhart polynomials of reflexive polytopes

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1. Introduction

$P \subset \mathbb{R}^N$: a lattice polytope of dimension d.
Suppose that $0 \in P \backslash \partial P$. Then the **dual polytope** is defined by

$$P^* := \{y \in \mathbb{R}^N \mid \langle x, y\rangle \le 1 \text{ for all } x \in P\}.$$

If P^* is a lattice polytope, then P is called a **reflexive polytope**.

Given $n \in \mathbb{Z}_{>0}$, we set

$$E(P, n) := \left|nP \cap \mathbb{Z}^N\right|,$$

where $nP = \{n\alpha \in \mathbb{R}^N : \alpha \in P\}$.

- $E(P, n)$ is a polynomial in n of degree d.

$E(P, n)$ is called the **Ehrhart polynomial** of P.

$$1 + \sum_{n\ge 1} E(P, n)t^n = \frac{\sum_{i=0}^{d} h_i^* t^i}{(1-t)^{d+1}}$$

We call $h^*(P, t) = \sum_{i=0}^{d} h_i^* t^i$ the **h^*-polynomial** of P.
If the roots of h^*-polynomial are real, then we say that h^*-polynomial is the **real-rooted**.

$$f(n) = \sum_{i=0}^{d} a_i n^i (1+n)^{d-i}.$$

If $a_0, \ldots, a_d \ge 0$, then we call $f(n)$ **magic positive**.

Theorem 1 ([2])
If $E(P, n)$ is magic positive, then h^*-polynomial is real-rooted.

- The Ehrhart polynomials of Zonotopes and Pitman-Stanley polytopes are magic positive.

Our goal is to present

a new polytope whose Ehrhart polynomial is magic positive !!

2. Stasheff Polytope

$\mathrm{St}_d := \mathrm{conv}(\{\pm \mathbf{e}_i : 1 \le i \le d\} \cup \{\mathbf{e}_i + \cdots + \mathbf{e}_j : 1 \le i < j \le d\})$.

St_d^* is called the **Stasheff polytope** (**associahedron**).

Proposition 2 ([1])
For $d \ge 2$, $E(\mathrm{St}_d^*, n)$ satisfies the following recurrence:

$$E(\mathrm{St}_d^*, n) = (2n+1)E(\mathrm{St}_{d-1}^*, n) - \frac{1}{2}n(n+1)E(\mathrm{St}_{d-2}^*, n)$$

Theorem 3 ([1])
$E(\mathrm{St}_d^*, n)$ is magic positive.

3. Dual of symmetric edge polytope

G : finite simple graph on $[d]$, with edge set $E(G)$.
The **symmetric edge polytope** $P_G \subset \mathbb{R}^d$ is defined by

$$P_G := \mathrm{conv}(\{\pm(e_v - e_w) \in \mathbb{R}^d \mid vw \in E(G)\})$$

Here, the vectors e_v are elements that form a lattice basis of $\mathbb{Z}^d$.

Proposition 4
T_d : a tree with d vertices
K_d : a complete graph with d vertices
$E(P_{T_d}^*, n)$ and $E(P_{K_d}^*, n)$ are magic positive.

For any connected graph G with d vertices, the following inclusions hold:

$$P_{K_d}^* \subset P_G^* \subset P_{T_d}^*$$

Question 5
G : connected finite simple graph
Is the $E(P_G^*, n)$ magic positive?

COUNTEREXAMPLES

○ : $E(P_{K_{a,b}}^*, n)$ is magic positive, × : $E(P_{K_{a,b}}^*, n)$ is not

a \ b	2	3	4	5	6	7	8	9
2	○	○	○	○	○	○	○	×
3	○	○	○	○	○	×	×	×
4	○	○	○	○	×	×	×	×
5	○	○	○	×	×	×	×	
6	○	○	×	×	×	×		
7	○	×	×	×	×			
8	○	×	×	×				
9	×	×	×					

Theorem 6 ([1])
C_d : a cycle of length d
We transform $E(P_{C_{d+1}}^*, n)$ into the form $E(P_{C_{d+1}}^*, n) = \sum_{j=0}^{d} a_j n^j (1+n)^{d-j}$. Then, the coefficients a_i and a_{d-i} are positive for $i = 0, 1, 2$.

REMARK By computational experiments, $E(P_{C_d}^*, n)$ is magic positive for all $d \le 500$.

Reference

[1] M. Konoike, A new class of magic positive Ehrhart polynomials of reflexive polytopes, arXiv:1107.4862v1.

[2] L. Ferroni and A. Higashitani, Examples and counterexamples in Ehrhart theory, *EMS Surveys in Mathematical Sciences*, 2024