On the Classification of Schubert Varieties in Partial Flag Varieties

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Notations

- G: a complex reductive Lie group
- S: the set of simple reflections
- $A = (a_{st})_{(s,t)\in S^2}$: Cartan matrix
- W: the Weyl group
- I: a subset of S
- $X(A, I) := G/P_I$: a flag variety
- $X(w, A, I)$: the closure of BwP/P, a Schubert variety
- $S(w) = \{s \in S | s \leq w\}$: the support of $w \in W$

Introduction

Schubert varieties form an extensively studied class of algebraic varieties whose properties are often characterized by combinatorics. We consider the following question, which was raised by Develin, Martin, and Reiner in [2].

Problem

Classify of all Schubert varieties up to algebraic isomorphism.

Using Cartan equivalence, Richmond and Slofstra solved this problem for Schubert varieties $X(w, A, \varnothing)$ in complete flag varieties $X(A, \varnothing) = G/B$ in [3]. We consider the general case.

Schubert Basis

The integral cohomology group $H^*(X(w, A, I))$ has a $\mathbb{Z}$-basis $\Sigma(w, A, I) = \sigma_v$ (called the *Schubert basis*), indexed by $v \in [1, w]^I$. It is determined by the variety structure of the Schubert variety.

Poset Isomorphism

The identification $i : [1, w]^I \to \Sigma(w, A, I)$, given by $v \mapsto \sigma_v$, is a poset isomorphism.

Under the above isomorphism, the image of the *right descent set* $D_R(v)$ for $v \in [1, w]^I$ is also an invariant.

Non-Equally Supported Pairs

From the poset isomorphism, we can deduce that the set $S(w) \setminus I$, as the inverse image of $H^2(X(w, A, I))$, is an invariant. If there is a Schubert variety $X(w', A', I')$ isomorphic to the previous one, then $|S(w) \setminus I| = |S(w') \setminus I'|$. If we further have $|S(w)| = |S(w')|$, then we say this pair of Schubert varieties is *equally supported*. The following proposition gives a simple condition when the non-equally supported pair is not isomorphic to each other.

Criterion for Non-Isomorphisms

If $S(w) \cap I$ is empty but $S(w') \cap I'$ is not, then $X(w, A, I)$ and $X(w', A', I')$ are not isomorphic as algebraic varieties.

When both $S(w)\setminus I$ and $S(w')\setminus I'$ are nonempty, there exist isomorphic pairs of Schubert varieties. The author constructed a large class of isomorphisms of Schubert varieties in flag varieties corresponding to maximal parabolic subgroups in [1].

Equally Supported Pairs

Nevertheless, the equally supported pairs are much easier to deal with. The following criterion generalizes the one given by Richmond and Slofstra in [3].

Criterion for Isomorphisms

Take $w \in W^I$ and $w' \in W'^{I'}$. If there is a bijection $\tau : S(w) \to S(w')$ sending $S(w) \cap I$ to $S(w') \cap I'$ such that

- for some reduced word $w = s_1 \cdots s_k$, $w' = \tau(s_1) \cdots \tau(s_k)$ is also a reduced word, and
- for any $t_1, t_2 \in S$, $a_{t_1 t_2} = a'_{\tau(t_1)\tau(t_2)}$ whenever $t_1 t_2 \leq w$,

then $X(w, A, I)$ and $X(w', A', I')$ are isomorphic.

Conversely, the above criterion is also necessary for the cases $|S(w) \cap I| = 0$ ([3, Theorem 1.3]) and $|S(w) \cap I| = 1$ ([1, Proposition 4.9]).

Enumeration of Isomorphism Classes

There are finitely many isomorphism classes of Schubert varieties in fixed dimension. Our criterion makes it possible to enumerate the isomorphism classes in low dimensions. Up to algebraic isomorphism,

- there is a unique Schubert curve;
- there are 7 Schubert surfaces;
- there are at most 34 Schubert three-folds.

Open Problems

- For equally supported pairs of Schubert varieties, is our sufficient condition for isomorphisms necessary?
- Are there examples of isomorphic, non-equally supported pairs of Schubert varieties *not* corresponding to maximal parabolic subgroups?
- Can one enumerate isomorphic classes of Schubert k-folds for $k \geq 3$?

References

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[2] Mike Develin, Jeremy L Martin, and Victor Reiner. “Classification of Ding’s Schubert varieties: finer rook equivalence”. In: *Canadian Journal of Mathematics* 59.1 (2007), pp. 36–62.

[3] Edward Richmond and William Slofstra. “The isomorphism problem for Schubert varieties”. In: *arXiv preprint arXiv:2103.08114* (2021).

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