The HOMFLY Polynomial of a Forest Quiver

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Introduction

Postnikov introduced plabic graphs in order to study a stratification of the totally nonnegative Grassmannian into positroid cells [3]. These graphs have proved to be useful intermediate objects in establishing connections between cluster algebras and knot theory. For example, in [1] Galashin and Lam studied connections between invariants of links and quivers which can be associated to these plabic graphs. We establish further connections by defining the HOMFLY polynomial of a forest quiver and relating it to the HOMFLY polynomial of certain plabic links [4].

Plabic links generalize the class of positroid links, which are links that can be associated to positroids or various objects in bijection with them. The invariants of these links in certain cases can provide information about associated objects such as positroid varieties.

The HOMFLY Polynomial of a Link

The *HOMFLY polynomial* is a link invariant which is a Laurent polynomial in the variables a and z. It is defined by the skein relation

$$aP(L_+) - a^{-1}P(L_-) = zP(L_0) \tag{1}$$

and setting $P(\text{Unknot}) = 1$ where L_+, L_-, and L_0 are links whose diagrams are the same except locally at one location where they are related as follows:

L_+ L_- L_0

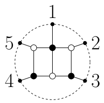
Example.

L_+ L_- L_0

$P(\text{2-component unlink}) = \frac{aP(\text{Unknot}) - a^{-1}P(\text{Unknot})}{z} = \frac{a-a^{-1}}{z}$.

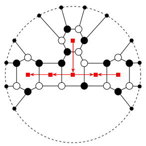
Plabic Graphs and Their Links and Quivers

A *plabic graph* G is a planar, bicolored graph which is embedded in the disk and whose vertices are colored black and white.



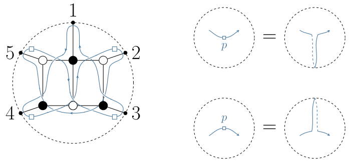
One can associate a directed, planar graph Q_G to each plabic graph G as follows:

- Place one vertex v_F inside each interior face F of G.
- Place an edge between vertices v_F and $v_{F'}$ for each edge e in G with opposite colored endpoints such that F and F' are adjacent to e.
- Orient the edge in Q_G so that as one travels along it, the white vertex in e is to the left.



We refer to Q_G as the *quiver* of G.

One can draw a link diagram for a plabic link L_G^{plab} of G by traveling along the edges in G, turning maximally right at each black vertex and maximally left at each white vertex. The crossings are determined according to the arguments of the tangent vectors in the complex plane. The strand with the smaller argument goes over the strand with the larger argument.



Motivations to Study Polynomial Invariants of Plabic Links

Theorem [2]. For an acyclic, mutable quiver Q with n vertices the point count over $\mathbb{F}_q$ of the associated cluster variety $\mathcal{A}$ is given by

$$\#\mathcal{A}(\mathbb{F}_q) = R(Q;q) = \sum_{i\geq 0} a_i(Q)(q-1)^{n-2i}q^i, \tag{2}$$

where $a_i(Q)$ is the number of independent sets of size i in the quiver Q.

Theorem [1]. Given a link L, let $P^{\text{top}}(L;q)$ be obtained from the top a-degree term of $P(L)$ via the substitutions $a = q^{-1/2}$ and $z = q^{1/2} - q^{-1/2}$. For leaf recurrent plabic graphs G, which include reduced plabic graphs, the equality $R(Q_G;q) = P^{\text{top}}(L_G^{\text{plab}};q)$ holds.

Guiding Question: *Can we go directly from the combinatorial objects, e.g. a plabic graph or its quiver, to the link invariants?*

The HOMFLY Polynomial of a Forest Quiver

Definition [4]. Let Q be a forest quiver. The *HOMFLY polynomial of* Q, denoted $f(Q)$, is defined recursively by setting

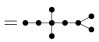
- $f(Q) = 1$ if Q is empty,
- $f(Q) = \frac{z+z^{-1}}{a} - \frac{z^{-1}}{a^3}$ if Q is a single vertex,
- $f(Q) = \frac{z}{a}f(Q-\{v\}) + \frac{1}{a^2}f(Q-\{v,\tilde{v}\})$ if v is a leaf in Q which is adjacent to $\tilde{v}$, and
- $f(Q) = f(Q_1)\cdot f(Q_2)$ if $Q = Q_1 \sqcup Q_2$.

The Alexander Polynomial $\Delta(Q)$ is obtained from $f(Q)$ by setting $a = 1$ and $z = t^{1/2} - t^{-1/2}$.

Connections Between the HOMFLY Polynomials

Theorem [4]. Let G be a connected plabic graph whose quiver Q_G is a forest quiver. Then $P(L_G^{\text{plab}}) = f(Q_G)$.

Example.

$Q =$ 

$$\Delta(Q) = t^{-9/2}(t^9 - t^8 - 3t^7 + 7t^6 - 8t^5 + 8t^4 - 7t^3 + 3t^2 + t - 1)$$

$$f(Q) = \frac{z^9+9z^7+28z^5+39z^3+28z+11z^{-1}+2z^{-3}}{a^9} - \frac{z^7+11z^5+36z^3+47z+28z^{-1}+7z^{-3}}{a^{11}} + \frac{5z^3+20z+23z^{-1}+9z^{-3}}{a^{13}} - \frac{z+6z^{-1}+5z^{-3}}{a^{15}} + \frac{z^{-3}}{a^{17}}$$

A Closed Formula

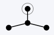
Let Q be a be a forest quiver with n vertices. Fix a root vertex in each component of Q. Given an independent set I in Q, let $p(I)$ be the number of vertices in Q which are the parent of at least one vertex in I.

Theorem [4]. Given a forest quiver Q with n vertices, the HOMFLY polynomial of Q is given by

$$f(Q) = \frac{1}{a^n}\left(\sum_{i,j\geq 0} c_{i,j}(Q)z^{n-2i}(1-a^{-2})^j\right), \tag{3}$$

where $c_{i,j}(Q)$ is the number of independent sets I of size i in Q such that $p(I) = i-j$.

Example. Consider the Dynkin diagram D_4. The only non-zero coefficients $c_{i,j}(D_4)$ are pictured below. We have chosen the top leaf to be the root. The vertices in each independent set are colored red, and their parent vertices are colored green.

$c_{0,0}(D_4) = 1$ 

$c_{2,1}(D_4) = 3$ 

$c_{1,0}(D_4) = 3$ 

$c_{3,2}(D_4) = 1$ 

$c_{1,1}(D_4) = 1$ 

Then the HOMFLY polynomial $f(D_4)$ is

$$f(D_4) = \frac{1}{a^4}\left(z^4 + 3z^2 + z^2(1-a^{-2}) + 3(1-a^{-2}) + z^{-2}(1-a^{-2})^2\right)$$
$$= \frac{z^4+4z^2+3+z^{-2}}{a^4} - \frac{z^2+3+2z^{-2}}{a^6} + \frac{z^{-2}}{a^8}.$$

Corollary. Let Q be a forest quiver with n vertices. The Alexander polynomial of Q is given by

$$\Delta(Q) = t^{-n/2}\cdot\sum_{i\geq 0} b_i(Q)(t-1)^{n-2i}t^i, \tag{4}$$

where $b_i(Q)$ is the number of ways to choose i distinct edges in Q which do not share any endpoints. Alternatively, $b_i(Q)$ is the number of independent sets of size i in the line graph of Q.

Remark. By substituting $a = q^{-1/2}$ and $z = q^{1/2} - q^{-1/2}$ into the top a-degree term of (3), we recover the formula for $R(Q;q)$ given in (2) in the case where Q is a forest quiver.

References

[1] Pavel Galashin and Thomas Lam. "Plabic links, quivers, and skein relations". In: *Algebraic Combinatorics* 7 (2024), pp. 431–474. DOI: 10.5802/alco.345.

[2] Thomas Lam and David Speyer. "Cohomology of cluster varieties. II. Acyclic case". In: *Journal of the London Mathematical Society* 108.6 (2023), pp. 2377–2414. DOI: 10.1112/jlms.12809.

[3] Alexander Postnikov. *Total positivity, Grassmannians, and networks*. 2006. arXiv: math/0609764 [math.CO]. URL: https://arxiv.org/abs/math/0609764.

[4] Amanda Schwartz. *The HOMFLY Polynomial of a Forest Quiver*. 2024. arXiv: 2410.00399 [math.CO]. URL: https://arxiv.org/abs/2410.00399.

Acknowledgments

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