Inversions in Parking Functions

Kyle Celano [1] Jennifer Elder [2] Kimberly P. Hadaway* [3] Pamela E. Harris [4] Amanda Priestley [5] Gabe Udell [6]

[1]Wake Forest University [2]Missouri Western State University [3]Iowa State University [4]University of Wisconsin-Milwaukee [5]The University of Texas at Austin [6]Cornell University

Abstract

In this poster, we obtain a q-exponential generating function for inversions on parking functions through a direct bijection to labeled rooted forests. Moreover, we obtain an expression for the total number of inversions across all parking functions via a probabilistic approach. Finally, by applying these techniques to *unit interval parking functions* (defined by Hadaway 2021) we give analogous results.

What is a parking function?

* A set of n cars want to park in n spots on a *one-way* street.
* Cars enter one at a time, and park in the first available spot *on or after* their preference. If they don't find a spot, then they drive away.
* If **all** cars park, then the set of preferences is a *parking function*. Let PF_n be the set of all parking functions of length n.

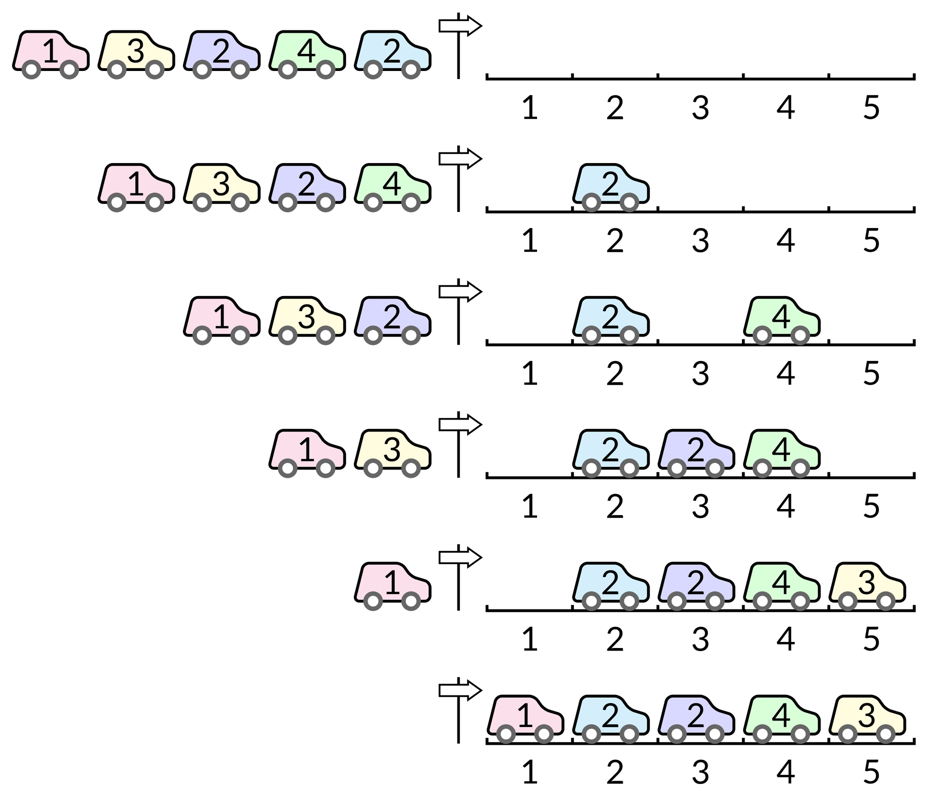


Figure 1. An example of the parking process with preferences $(2, 4, 2, 3, 1)$.

Other definitions

* The *outcome permutation* $\pi(\alpha)$ is defined by setting $\pi(\alpha)(j) = i$ if car i parks in spot j.
* A *unit interval parking function* of length n is a parking function $\alpha \in \mathrm{PF}_n$ such that $\pi(\alpha)^{-1}(i) - \alpha_i \leq 1$ for all $i \in [n]$. Let UPF_n denote the set of unit interval parking functions of length n.
* For a word $w \in \mathbb{P}^n$, an *inversion* is a pair (i, j) of integers in $[n]$ such that $i < j$ and $w_i > w_j$. We denote the set of inversions of a word w by $\mathrm{Inv}(w)$ and let $\mathrm{inv}(w) = |\mathrm{Inv}(w)|$.

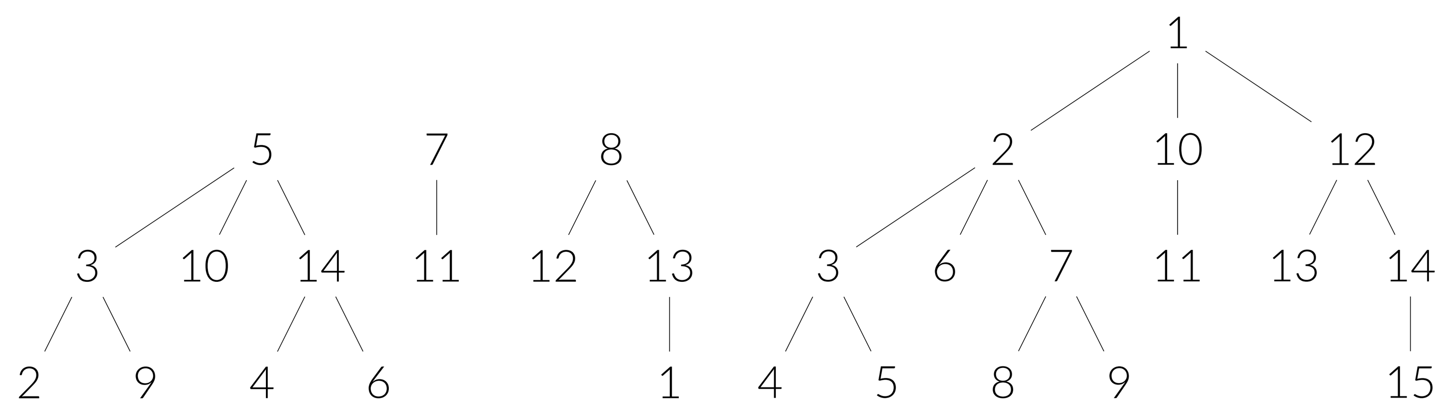
Labeled rooted forests

* A *labeled rooted forest* is a rooted forest is made up of rooted trees in which every vertex is given a unique integer.
* The *subtrees* of a labeled rooted tree T are labeled rooted subtrees $T_1, T_2, \ldots, T_k$ such that $r(T_i)$ is adjacent in T to $r(T)$ for each $i \in [k]$, where we order the trees so that $r(T_1) < r(T_2) < \cdots < r(T_k)$.
* Let T be a labeled rooted tree with root r. The *preorder traversal permutation* $w(T)$ of T is defined recursively by setting

$$w(T) = \begin{cases} r(T) & \text{if } T \text{ is a single vertex } r(T) \\ r(T) \cdot w(T_1) w(T_2) \cdots w(T_k) & \text{if } T \text{ has subtrees } T_1, T_2, \ldots, T_k, \end{cases}$$

where $u \cdot v$ denote concatenation of words u and v.

* Suppose $F \in \mathcal{F}_n$. The pair (i, j) of integers $i, j \in [n]$ is called a parental preorder inversion of F provided $i < j$ and $w_F^{-1}(p(i)) > w_F^{-1}(p(j))$. We denote the number of parental preorder inversions of F by $\mathrm{pinv}(F)$.



(a) Labeled rooted forest F (b) Preorder traversal of $T(F)$

Figure 2. $w(F) = (0, 5, 3, 2, 9, 10, 14, 4, 6, 7, 11, 8, 12, 13, 1)$
$\rho(F) = (14, 3, 2, 7, 1, 7, 1, 1, 3, 2, 10, 12, 12, 2)$

Proposition

For all $n \geq 1$, $\mathrm{PF}_n(q) = \sum_{F \in \mathcal{F}_n} q^{\mathrm{pinv}(F)}$.

Future work

* For other families of parking functions, find nice expressions for their inversion generating functions. Furthermore, determine if there is a natural $\mathfrak{S}_n$-action on the family and determine its Frobenius character.
* Investigate generating functions for other word statistics on parking functions and their subsets and generalizations.

Total number of inversions

Theorem

Let $W \subset \mathbb{P}^n$ be an $\mathfrak{S}_n$-invariant set of words of positive integers. Then, we have the following expectations:

(a) $\mathbb{E}_W[\mathrm{inv}] = \binom{n}{2} \mathbb{P}_W(\mathrm{des}_1 = 1)$,
(b) $\mathbb{E}_W[\mathrm{des}] = (n-1)_W(\mathrm{des}_1 = 1)$, and
(c) $\mathbb{E}_W[\mathrm{inv}] = \frac{n}{2} \mathbb{E}_W[\mathrm{des}]$.

Theorem (Schumacher)

For all $n \geq 1$, we have that

$$\sum_{\alpha \in \mathrm{PF}_n} \mathrm{des}(\alpha) = \binom{n}{2} (n+1)^{(n-2)}.$$

Corollary

For $n \geq 1$, the total number of inversions across all parking functions is

$$\sum_{\alpha \in \mathrm{PF}_n} \mathrm{inv}(\alpha) = \frac{n(n+1)^{n-2}}{2} \binom{n}{2}.$$

Let $(\mathrm{Fub}_n)_{n \geq 1}$ denote the Fubini numbers.

Theorem

For $n \geq 1$, we have

$$\sum_{\alpha \in \mathrm{UPF}_n} \mathrm{des}(\alpha) = \sum_{\alpha \in_n} \mathrm{des}(\alpha) = \frac{n-1}{2} (\mathrm{Fub}_n - \mathrm{Fub}_{n-1}).$$

Corollary

For $n \geq 1$, we have

$$\sum_{\alpha \in \mathrm{UPF}_n} \mathrm{inv}(\alpha) = \sum_{\alpha \in_n} \mathrm{inv}(\alpha) = \frac{n(n-1)}{4} (\mathrm{Fub}_n - \mathrm{Fub}_{n-1}).$$

More Information

Find paper posted soon on ArXiv or my website: www.kimberlyphadaway.com

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