The peak algebra in noncommuting variables

Farid Aliniaeifard and Shu Xiao Li
farid@sdu.edu.cn, lishuxiao@sdu.edu.cn
SHANDONG UNIVERSITY

Introduction

The well-known descent-to-peak map Θ_{QSym} for the Hopf algebra of quasisymmetric functions, QSym, and the peak algebra Π were originally defined by Stembridge in 1997. We define the labelled descent-to-peak map Θ_{NCQSym} and extend the notion of the peak algebra to noncommuting variables. Moreover, we define Schur Q-functions in noncommuting variables having properties analogous to the classical Schur Q-functions.

Generalized chromatic functions

A *labelled edge-coloured digraph* is a digraph with two types of edges $\rightarrow$ and $\Rightarrow$ where its vertex set is a subset of $\mathbb{N}$. A *proper* colouring of a labelled edge-coloured digraph $\mathbf{G}$ is a function

$$\kappa : V(\mathbf{G}) \rightarrow \mathbb{N} = \{1, 2, 3, \dots\}$$

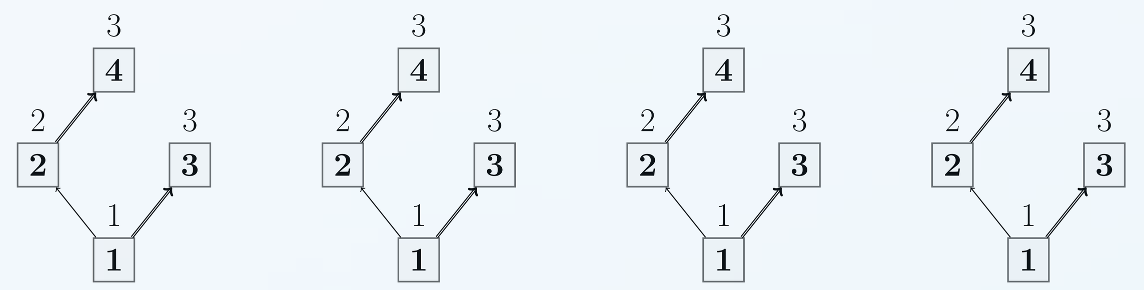
such that

1. If $a \Rightarrow b$, then $\kappa(a) \leq \kappa(b)$.
2. If $a \rightarrow b$, then $\kappa(a) < \kappa(b)$.

The *generalized chromatic function in noncommuting variables* of a labelled edge-coloured digraph $\mathbf{G}$ with vertex set $[n]$ is

$$\mathscr{Y}_{\mathbf{G}} = \sum_{\kappa} \mathbf{x}_{\kappa(1)}\mathbf{x}_{\kappa(2)} \cdots \mathbf{x}_{\kappa(n)}$$

where the sum is over all proper colourings κ of $\mathbf{G}$.



$$\mathscr{Y}_G = \mathbf{x}_1\mathbf{x}_2\mathbf{x}_2\mathbf{x}_2 + \mathbf{x}_1\mathbf{x}_2\mathbf{x}_2\mathbf{x}_3 + \mathbf{x}_1\mathbf{x}_2\mathbf{x}_3\mathbf{x}_2 + \mathbf{x}_1\mathbf{x}_2\mathbf{x}_3\mathbf{x}_3 + \cdots$$

The span of the set $\{\mathscr{Y}_{\mathbf{G}}\}$ gives the Hopf algebra of *quasisymmetric functions in noncommuting variables,* NCQSym, and the space of symmetric functions in NCQSym is called the Hopf algebra of *symmetric functions in noncommuting variables,* NCSym. For any set partition π of n, the *power sum symmetric function in noncommuting variables* is

$$\mathbf{p}_\pi = \sum_{\substack{i_j = i_k \\ \text{if } j,k \text{ in the same block of } \pi}} \mathbf{x}_{i_1}\mathbf{x}_{i_2}\cdots\mathbf{x}_{i_n},$$

and the *elementary symmetric function in noncommuting variables* is

$$\mathbf{e}_\pi = \sum_{\substack{i_j \neq i_k \\ \text{if } j,k \text{ in the same block of } \pi}} \mathbf{x}_{i_1}\mathbf{x}_{i_2}\cdots\mathbf{x}_{i_n}.$$

Enriched chromatic functions

Given an labelled edge-coloured digraph $\mathbf{G}$, an *enriched* colouring of $\mathbf{G}$ is a function

$$\kappa : V(\mathbf{G}) \rightarrow \{-1 \prec 1 \prec -2 \prec 2 \prec \cdots\}$$

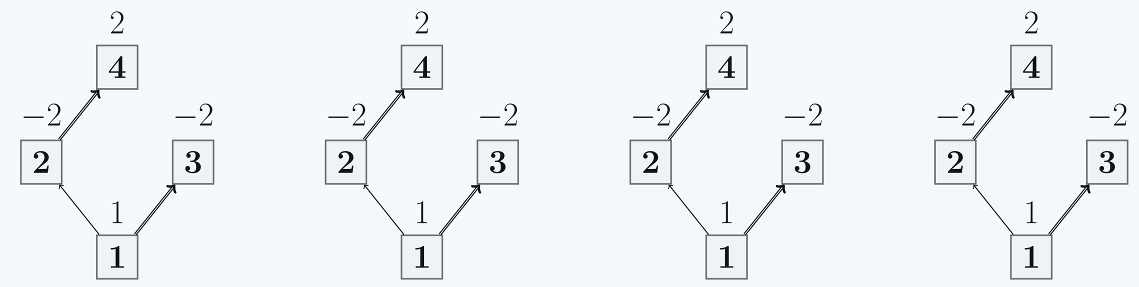
such that

1. If $a \Rightarrow b$, then either $\kappa(a) \prec \kappa(b)$ or $\kappa(a) = \kappa(b) > 0$.
2. If $a \rightarrow b$, then either $\kappa(a) \prec \kappa(b)$ or $\kappa(a) = \kappa(b) < 0$.

The *enriched chromatic function in noncommuting variables* of a labelled edge-coloured digraph $\mathbf{G}$ with vertex set $[n]$ is

$$\mathscr{F}_{\mathbf{G}} = \sum_{\kappa} \mathbf{x}_{|\kappa(1)|}\mathbf{x}_{|\kappa(2)|} \cdots \mathbf{x}_{|\kappa(n)|}$$

where the sum is over all enriched colouring κ of G.



$$\mathscr{F}_{\mathbf{G}} = \mathbf{x}_1\mathbf{x}_1\mathbf{x}_1\mathbf{x}_1 + \mathbf{x}_1\mathbf{x}_1\mathbf{x}_1\mathbf{x}_1 + \mathbf{x}_1\mathbf{x}_2\mathbf{x}_1\mathbf{x}_2 + \mathbf{x}_1\mathbf{x}_2\mathbf{x}_2\mathbf{x}_2 + \cdots$$

The span of the set $\{\mathscr{F}_{\mathbf{G}}\}$ gives the *peak algebra in noncommuting variables,* NCΠ.

Labeled descent-to-peak map

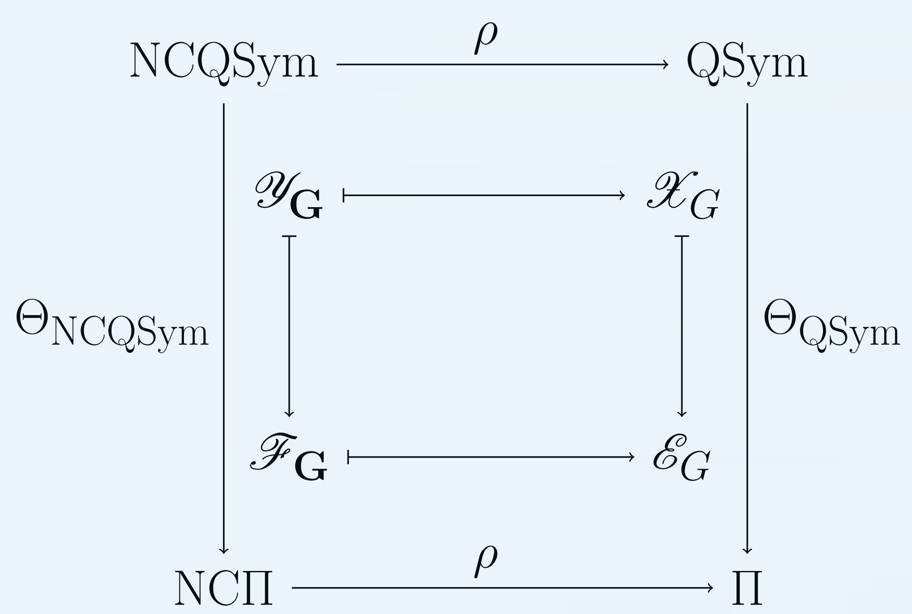
The map

$$\begin{aligned} \Theta_{\mathrm{NCQSym}} : \mathrm{NCQSym} &\rightarrow \mathrm{NC}\Pi \\ \mathscr{Y}_{\mathbf{G}} &\mapsto \mathscr{F}_{\mathbf{G}} \end{aligned}$$

is well-defined and is called the *labelled descent-to-peak map*.

Properties

- We have $\dim(\mathrm{NC}\Pi_n) = |\{\phi \vDash [n] : \phi \text{ is an odd set composition}\}|$.
- NCΠ is a Hopf algebra.
- The labelled descent-to-peak map, Θ_{NCQSym}, is a diagonalizable surjective Hopf algebra morphism, and we have



- Define $\Theta_{\mathrm{NCSym}} = \Theta_{\mathrm{NCQSym}}|_{\mathrm{NCSym}}$. For set partition $\pi = \pi_1/\pi_2/\cdots/\pi_{\ell(\pi)} \vdash [n]$, we have

$$\Theta_{\mathrm{NCSym}}(\mathbf{p}_\pi) = \begin{cases} 2^{\ell(\pi)}\mathbf{p}_\pi & \text{if all blocks of } \pi \text{ have odd sizes,} \\ 0 & \text{otherwise.} \end{cases}$$

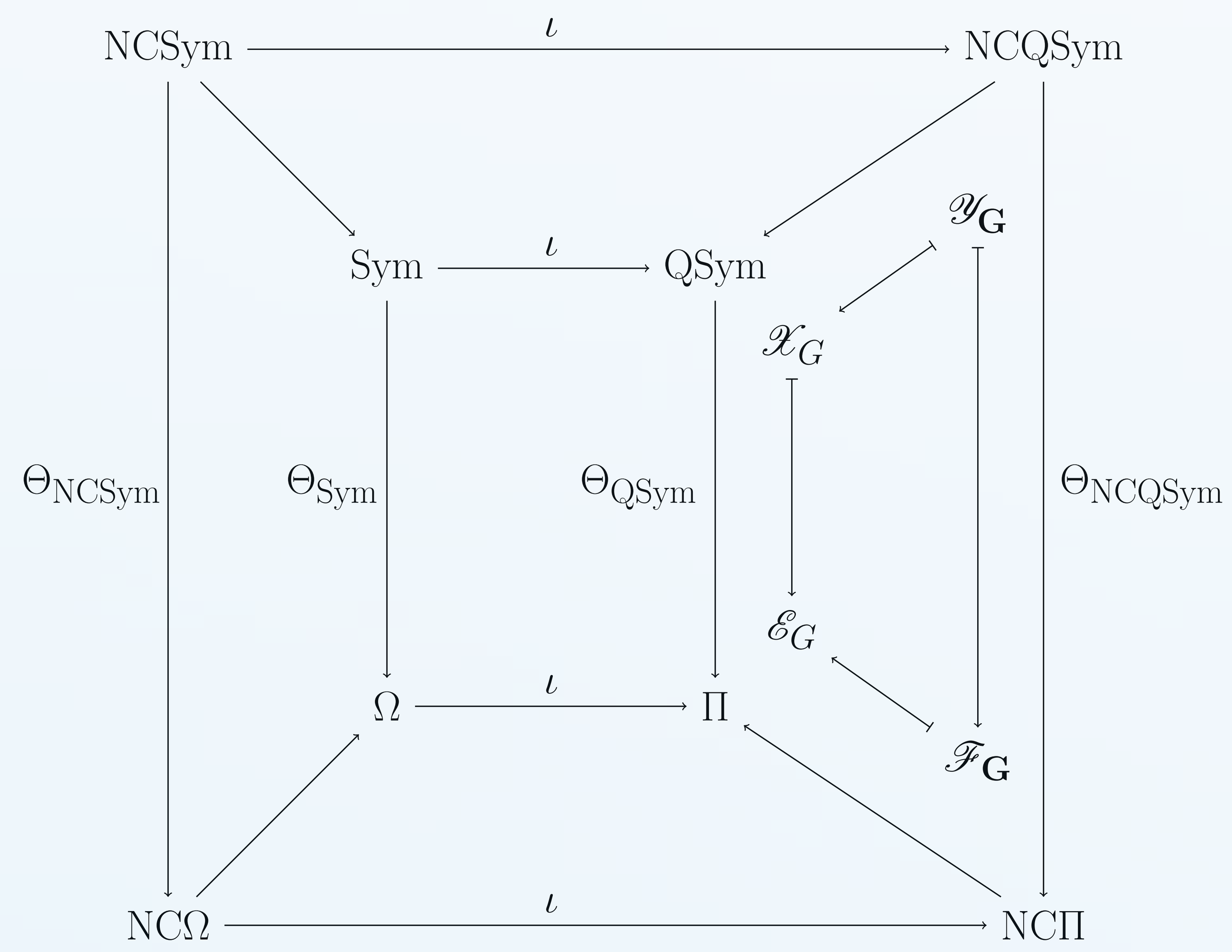
- For an *odd set partition* π, a set paritions whose all blocks have odd sizes, the *Schur Q-function in noncommuting variables* is

$$\mathbf{q}_\pi = \Theta_{\mathrm{NCSym}}(\mathbf{e}_\pi).$$

The Hopf algebra of *Schur Q-functions in noncommuting variables* is NC$\Omega = \mathbb{C}$-span$\{\mathbf{q}_\pi : \pi$ is an odd set partition$\}$.

$$\mathrm{NC}\Omega = \Theta_{\mathrm{NCQSym}}(\mathrm{NCSym}) \quad \text{and} \quad \mathrm{NC}\Omega = \mathrm{NC}\Pi \cap \mathrm{NCSym}$$

- We have the commutative diagram



- NCΠ is a left co-ideal in NCQSym under the internal comultiplication given by $M_\phi = \sum_{\psi \wedge \varphi = \phi} M_\psi \otimes M_\varphi$.

References

[1] F. Aliniaeifard and S. Li, *The peak algebra in noncommuting variables*, arXiv preprint arXiv:2506.12868, 2025.

[2] J. Stembridge, *Enriched P-partitions*, Transactions of the American Mathematical Society, 249: 763–788, 1997.