Fundamental groups of moduli spaces of real weighted stable curves

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Summary

- Hassett's space of weighted stable curves $\overline{M_{0,\mathcal{A}}}$: a blowdown of $\overline{M_{0,n+1}}$.
- We show how to tile $\overline{M_{0,\mathcal{A}}}(\mathbb{R})$ by **products of permutahedra** (Figures 1-3)
- When $\mathcal{A}$ is S_n-symmetric: we describe the ordinary and S_n-equivariant fundamental groups of $\overline{M_{0,\mathcal{A}}}(\mathbb{R})$, called **weighted cactus groups**
  - Prior work [1]: tiling of $\overline{M_{0,n+1}}(\mathbb{R})$ by cubes, presentation of (unweighted) cactus group

Main results. Let $\mathcal{A} = (a_1, \ldots, a_{n+1})$ be a weight vector with $a_{n+1} = 1$.
(1) $\overline{M_{0,\mathcal{A}}}(\mathbb{R})$ has a **cell decomposition** indexed by rooted trees τ, compatible with the blowdown maps. Each cell $\overline{W_\tau}$ is a product of permutahedra.
(2) Suppose $\mathcal{A} = \mathcal{A}(a) := (\frac{1}{a}, \ldots, \frac{1}{a}, 1)$ and $a \geq 3$. The weighted cactus group J_n^a is obtained from the unweighted cactus group by deleting generators and adding **braid relations**.

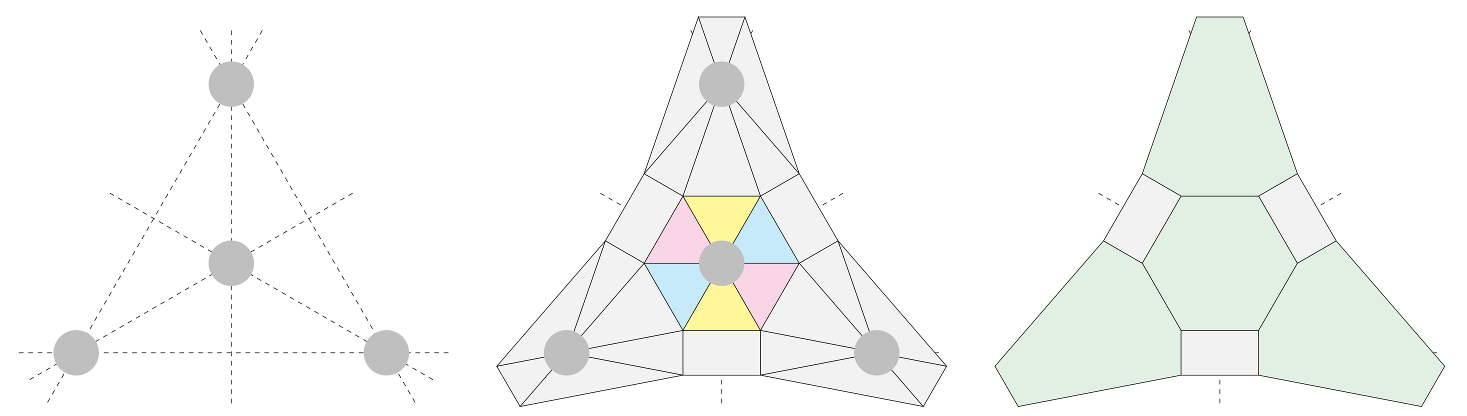


Figure 1: Left: $\overline{M_{0,5}}(\mathbb{R})$ is the blowup of $\mathbb{RP}^2$ at four points. **Center:** Decomposition into 15 regions homeomorphic to squares. **Right:** Blowing down and merging cells of $\overline{M_{0,\mathcal{A}(3)}}(\mathbb{R}) \cong \mathbb{RP}^2$ into permutahedra (hexagons).

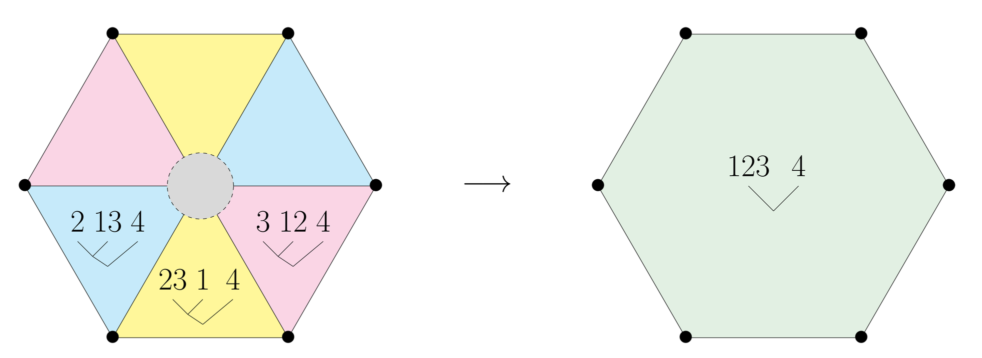


Figure 2: Detail of Figure 1, a local picture of the blowdown $\overline{M_{0,\mathcal{A}(2)}}(\mathbb{R}) \to \overline{M_{0,\mathcal{A}(3)}}(\mathbb{R})$, merging three regions homeomorphic to squares into a hexagon Π_3. Each region is labeled by a 2-stable or 3-stable tree. Note: on the inner circle, antipodal points are identified.

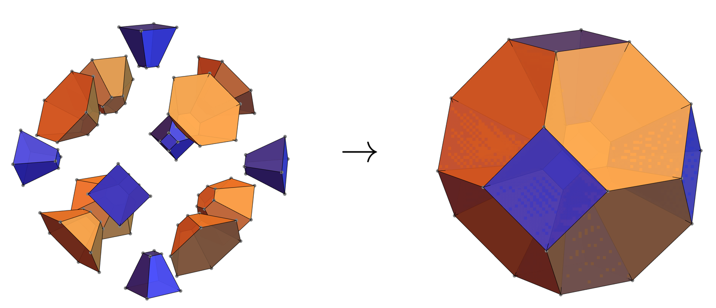


Figure 3: Analog of Figure 2 in three dimensions, blowing down three hexagonal prisms $\Pi_2 \times \Pi_3$ and three cubes $\Pi_2 \times \Pi_2 \times \Pi_2$ into a Π_4.

Rooted trees

An a**-stable rooted tree** τ on the set $[n]$ is a rooted tree with

(i) leaves labeled by subsets $A_i \subseteq [n]$ with $|A_i| \leq a$, forming a set partition of [n];

(ii) for each $v \in \tau$, an ordering of the children of v, up to reversal,

such that

(iii) for each $v \in \tau$, the union of all A_i in the subtree rooted at v has $> a$ elements.

If $b > a$, the **compression** of τ is the b-stable tree $\varpi_{b,a}(\tau)$ given by replacing each subtree violating (iv) by the union of its leaves.

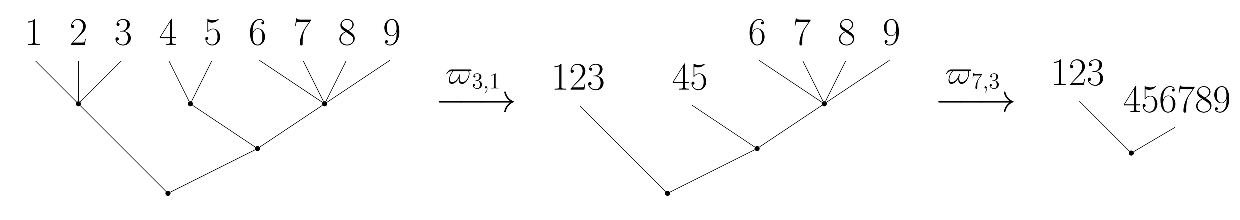


Figure 4: Compressing a 1-stable tree to a 3-stable tree, then to a 7-stable tree.

Permutahedra

- For each n, the **permutahedron** is $\Pi_n := \mathrm{conv}(\sigma : \sigma \in S_n) \subseteq \mathbb{R}^n$.
- Faces of Π_n are indexed by **ordered set partitions** of $[n]$. The face corresponding to $A_\bullet = (A_1, \ldots, A_k)$ is $\Pi_{A_\bullet} \cong \Pi_{|A_1|} \times \cdots \times \Pi_{|A_k|}$.

Moduli spaces of stable curves

- Let $\mathcal{A} = (a_1, \ldots, a_{n+1}) \in (0,1]^{n+1}$ with $a_{n+1} = 1$
- An $\mathcal{A}$**-stable** $(n+1)$**-marked curve of genus zero** is a union of copies of $\mathbb{P}^1$ with $n+1$ labeled points, satisfying certain "stability" conditions.

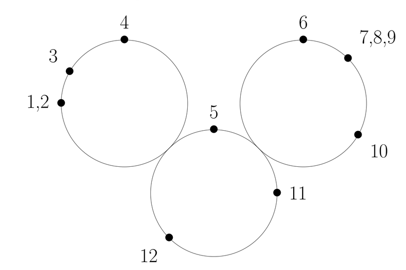


Figure 5: An $\mathcal{A}$-stable curve, where $\mathcal{A} = \mathcal{A}(3) = (\frac{1}{3}, \ldots, \frac{1}{3}, 1)$.

- Hassett's moduli space $\overline{M_{0,\mathcal{A}}}(\mathbb{R})$ parametrizes isomorphism classes of $(C; x_\bullet)$ with real marked points and nodes. It is a blowdown of $\overline{M_{0,n+1}}(\mathbb{R})$ and a blowup of $\mathbb{RP}^{n-2}$.

Distance algorithm and constructing the cell decomposition (1)

We associate to each $(C; x_\bullet) \in \overline{M_{0,\mathcal{A}}}(\mathbb{R})$ a **tree of distances** $\tau^{\mathrm{dist}}(C; x_\bullet)$. This decomposes the moduli space into (locally closed) cells

$$W_\tau = \{(C; x_\bullet) : \tau^{\mathrm{dist}}(C; x_\bullet) = \tau\}.$$

Distance algorithm (see Figure 6; cf. [2]).

1. If $C = \mathbb{P}^1$, set $x_{n+1} = \infty$. Join up all nearest neighbors. Repeat until all neighbors are joined.
2. (If C has multiple components, repeat on each component.)
3. Compress the resulting tree to an a-stable tree.

Cell decomposition of $\overline{M_{0,\mathcal{A}}}(\mathbb{R})$. Suppose τ has leaves labeled $A_1, \ldots, A_k$ and e internal edges. Then the (closure of the) cell corresponding to τ has the form

$$\overline{W_\tau} \cong [-1,1]^e \times \Pi_{|A_1|} \times \cdots \times \Pi_{|A_k|},$$

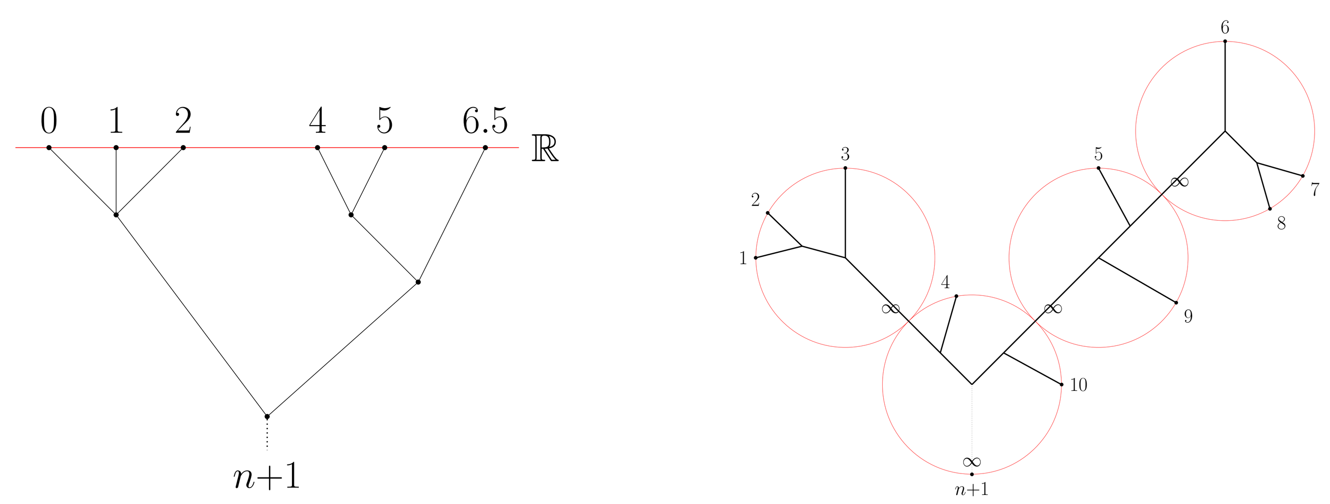


Figure 6: Left: 1-stable tree of distances from $x_\bullet = (0, 1, 2, 4, 5, 6.5, \infty)$ on $C = \mathbb{RP}^1$. **Right:** Tree of distances on a curve with 4 components.

Equivariant fundamental groups and proof of (2)

Let G be a finite group acting on X a space, $x_0 \in X$. The G**-equivariant fundamental group** is

$$\pi_1^G(X, x_0) = \{(g, \gamma) : g \in G, \gamma \text{ a homotopy class of paths from } x_0 \text{ to } gx_0\},$$

with group law $(g, \gamma) * (g', \gamma') = (gg', \gamma * g\gamma')$. There is a short exact sequence

$$1 \to \pi_1(X, x_0) \to \pi_1^G(X, x_0) \to G \to 1.$$

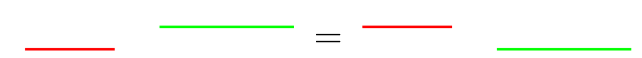
For $\overline{M_{0,n+1}}(\mathbb{R})$, S_n acts by permuting the marked points, fixing x_{n+1}. The **cactus group**

$$J_n = \pi_1^{S_n}(\overline{M_{0,n+1}}(\mathbb{R}))$$

is generated by "interval-reversing" elements $s_{p,q}$ for $1 \leq p < q \leq n$, with **cactus relations**:

(i) (involutions) $s_{p,q}^2 = 1$ for all p, q,

(ii) (disjoint intervals) $s_{p,q} s_{k,l} = s_{k,l} s_{p,q}$ when $p < q < k < l$:



(iii) (nested intervals) $s_{p,q} s_{k,l} = s_{p+q-l,p+q-k} s_{p,q}$ when $p \leq k < l \leq q$:

Presentation of the weighted cactus group. Let $\mathcal{A} = (\frac{1}{a}, \ldots, \frac{1}{a}, 1)$ with $a \geq 3$. The weighted cactus group $J_n^a := \pi_1^{S_n}(\overline{M_{0,\mathcal{A}}}(\mathbb{R}))$ has the generating set

$$\{s_{p,q} : 1 \leq p < q \leq n \text{ and } q - p \geq a \text{ or } q = p + 1\},$$

with relations given by the cactus relations (i)-(iii) above, along with
(iv) (**braid relations**) $(s_{p,p+1} s_{p+1,p+2})^3 = 1$ for all p.

Proof. The 2-cells of the permutahedral tiling are all

$$[-1,1]^2, [-1,1] \times \Pi_2, \Pi_2 \times \Pi_2, \text{ or } \Pi_3.$$

The first three are squares and induce the (four-term) cactus relations. The hexagon Π_3 induces the (six-term) braid relation (Figure 7). □

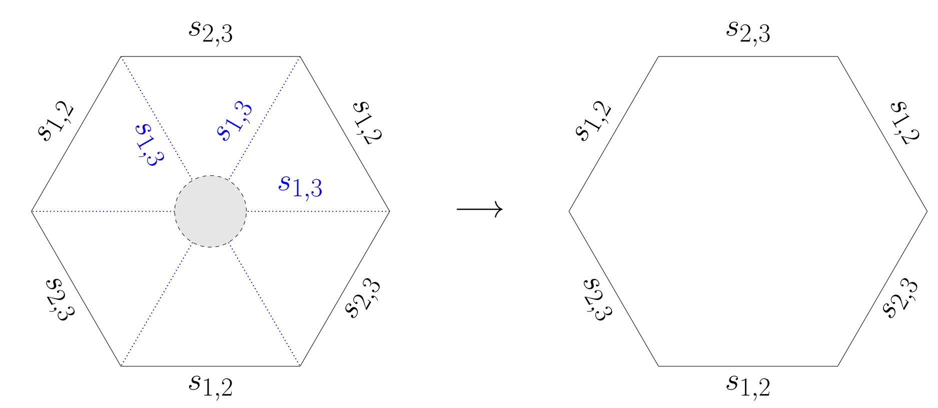


Figure 7: Showing how merging cells eliminates $s_{1,3}$ and introduces a braid relation.

References

[1] M. Davis, T. Januszkiewicz, and R. Scott, *Fundamental groups of blow-ups*, Adv. Math. **177** (2003), no. 1, 115–179. MR 1985196

[2] Aleksei Ilin, Joel Kamnitzer, Yu Li, Piotr Przytycki, and Leonid Rybnikov, *The moduli space of cactus flower curves and the virtual cactus group*, 2024, arXiv:2308.06880.

[3] Jake Levinson and Haggai Liu, *Fundamental groups of moduli spaces of real weighted stable curves*, 2025, arXiv:2503.23253.