Centralizers in the plactic monoid

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Notation

- $\mathbb{P} = \{1, 2, 3, \dots\}$, $\mathbb{N} = \mathbb{P} \uplus \{0\}$, and $\mathbb{P}^*$ is the set of words of positive integers.
- Given a row R of a tableau and a condition I on integers we let

$$R(I) = \text{ multiset of elements of } R \text{ satisfying } I.$$

- Given words u, w, let $u \equiv w$ denote that u and w are Knuth equivalent [1].
- Given a word w, let $P(w)$ be the RSK insertion tableau of w.

Centralizer of u

Given a word $u \in \mathbb{P}^*$, our primary object of study will be the *centralizer of* u in the plactic monoid which is

$$C(u) = \{w \mid uw \equiv wu\},$$

or equivalently

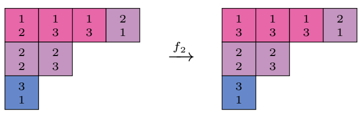
$$C(u) = \{w \mid P(uw) = P(wu)\}.$$

In particular, we wish to characterize $C(u)$ for certain u and also consider the enumerative properties of the integers

$$c_{n,m}(u) = \#\{w \in C(u) \mid \#w = n \text{ and } \max w \leq m\}.$$

Crystal motivation

This project was motivated in part by work of the second author and Nate Harman about a $\mathfrak{gl}_m \times \mathfrak{gl}_n$-crystal structure on lexicographic bitableaux with entries in $[m] \times [n]$. See below an example of how such a $\mathfrak{gl}_n$-crystal operator would act by raising the second entry of a box by extracting a reading word.



Bitableau Preprint

The goal is then a $\mathfrak{gl}_m$-crystal structure that modifies the first entry of a box in a way that commutes with the $\mathfrak{gl}_n$-crystal structure. Changing the first entry will move a letter in a reading word to another location within the word, and the crystal structures commute if the new reading word is Knuth equivalent to the old. The question of when moving a single letter in a word wouldn't change the Knuth class inspired this project on plactic monoid centralizers.

General strategy

Our principal tool is to compare the computation of $P(wu)$ using RSK with the computation of $P(uw)$ using jdt. In the former, the elements of u are inserted into $P(w)$ using the usual RSK bumping procedure. In the latter, a skew tableau is formed with $P(u)$ in the southwest and $P(w)$ in the northeast. The tableau is then brought to left-justified shape using jdt slides.

Lemma 1. Let $a \neq b$ be distinct positive integers and let $w \in \mathbb{P}^*$. Then

$$\alpha_b(P(wa)) \preceq \alpha_b(P(w)) \preceq \alpha_b(P(aw)).$$

where $\alpha_b(P)$ is a weak composition recording the number of instances of b in each row of P.

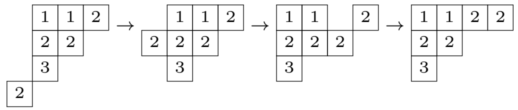


Figure 1. Jeu de taquin slides yielding the same result as RSK-insertion of a 2 demonstrate that $322112 \in C(2)$ (as well as all words Knuth equivalent to this word).

$|u| = 1$

When $|u| = 1$, we can completely characterize elements w of the centralizer $C(u)$ in terms of their insertion tableaux $P(w)$.

Theorem 1. Suppose u consists of a single integer which we also denote by u. Also, use $R_1, R_2, \dots, R_l$ to denote the rows of $P = P(w)$. Then the set $C(u)$ is all w such that $P = P(w)$ satisfies

(a) $\max R_1 \leq u$, and
(b) for $i \geq 1$ we have

$$\#R_i(< u) = \#R_{i+1}(\leq u).$$

Theorem 2. If $|u| = 1$ then

$$C(u) = \{w \mid \text{every column of } P = P(w) \text{ contains a } u\}.$$

$|u| = 2$ or $|u| = 3$

Theorem 3. We have that $w \in C(12)$ if and only if all columns C of $P(w)$ satisfy the following two conditions.

(a) If there is a singleton column, C, then C is a singleton 1-column or a singleton 2-column, and both types of columns must exist.
(b) If $\#C \geq 2$ then C must contain both 1 and 2.

Theorem 4. We have that $w \in C(212)$ if and only if all columns C of $P(w)$ satisfy the following two conditions.

(a) All singleton columns are singleton 2-columns.
(b) If $\#C \geq 2$ then C must contain both 1 and 2.

Longer u

We can show that, interestingly, when u consists of a single element a, the centralizer $C(a^k)$ does not depend on k.

Theorem 5. If $a, k \in \mathbb{P}$ then

$$C(a^k) = C(a).$$

There is another class of words that have a particularly nice characterization of their centralizers.

Theorem 6. We have $w \in C(m(m-1)\dots 1)$ if and only if $P = P(w)$ satisfies

$$\max R_i \leq m \quad \text{for all } 1 \leq i \leq m$$

where R_i is the ith row of P.

Enumeration

Using Stanley's theory of $\mathfrak{P}$-partitions (see [2, Section 7.4] or [3, Section 3.15]), we show that under certain conditions, $c_{n,m}(u)$ is given by a polynomial.

Theorem . If $n \geq k$ then $c_{n,m}(k(k-1)\dots 1)$ is a polynomial in m of degree $n-k$ with leading coefficient $1/(n-k)!$.

Theorem . Suppose n is fixed and $m \geq n$. Then we have the following polynomial expansions.

$$c_{2,m}(1) = \binom{m}{1},$$
$$c_{3,m}(1) = \binom{m}{1} + \binom{m}{2},$$
$$c_{4,m}(1) = \binom{m}{1} + 4\binom{m}{2} + \binom{m}{3},$$
$$c_{5,m}(1) = \binom{m}{1} + 8\binom{m}{2} + 13\binom{m}{3} + \binom{m}{4},$$
$$c_{6,m}(1) = \binom{m}{1} + 18\binom{m}{2} + 48\binom{m}{3} + 41\binom{m}{4} + \binom{m}{5},$$
$$c_{7,m}(1) = \binom{m}{1} + 33\binom{m}{2} + 178\binom{m}{3} + 262\binom{m}{4} + 131\binom{m}{5} + \binom{m}{6},$$
$$c_{8,m}(1) = \binom{m}{1} + 68\binom{m}{2} + 549\binom{m}{3} + 1480\binom{m}{4} + 1405\binom{m}{5} + 428\binom{m}{6} + \binom{m}{7}.$$

(Note that Catalan -1 appears on the diagonal)

Open problems and conjectures

Stability conjecture. Suppose $u \in \mathbb{P}^*$.

(a) There is a $K \in \mathbb{P}$ such that for $k \geq K$ we have

$$C(u^k) \subseteq C(u^{k+1}).$$

(b) There is a $L \in \mathbb{P}$ such that for $k \geq L$ we have

$$C(u^k) = C(u^{k+1}).$$

We have verified (a) computationally using Sage Math for $u \in [m]^n$ and $w \in [5]^l$ where $m + n \leq 10$ and $2 \leq l \leq 6$. Note that except in the particular case that $u = 12345$ where $K = 3$, for all other words u checked, we can take $K = 1$. In support of (b), the containments verified under these conditions become equalities for $k \geq 4$.

Unimodality conjecture. Fix n and write

$$c_{n,m}(1) = \sum_{k=0}^{n-1} a_k \binom{m}{k}$$

for certain scalars a_k (depending on n). We have the following

(a) $a_0 = 0, a_1 = 1$.
(b) $a_k \in \mathbb{P}$ for all $k \in [n-1]$.
(c) The sequence $a_1, a_2, \dots, a_{n-1}$ is log-concave and hence (assuming (b)) unimodal with maximum at $k = \lceil n/2 \rceil$.

References

[1] Donald E. Knuth. Permutations, matrices, and generalized Young tableaux. *Pacific J. Math.*, 34:709–727, 1970.

[2] Bruce E. Sagan. *Combinatorics: the art of counting*, volume 210 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, [2020] ©2020.

[3] Richard P. Stanley. *Enumerative combinatorics. Volume 1*, volume 49 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2012.