A geometric interpretation and skewing formula for the Delta Theorem

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Abstract

Our main results are:

- A **skewing identity** directly relating the **Rational Shuffle Theorem** concerning $E_{K,k}\cdot 1$ to the **Delta Theorem** concerning $\Delta'_{e_{k-1}}e_n$.
- Both a combinatorial and an algebraic proof of the skewing identity.
- A **geometric interpretation** of both symmetric functions as the bigraded S_n action on the homology of a variety.

Background: A tale of two operators

Give combinatorial formulas for the evaluations of two complicated operators on symmetric functions $\mathrm{Sym}_{q,t}$:

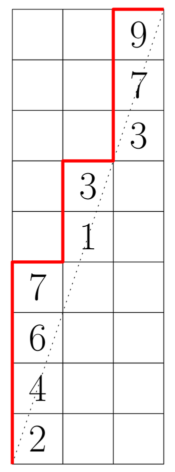
- $E_{kn,km}$ raises the degree of a symm. function by kn, called an **elliptic Hall algebra operator**.
- $\Delta'_{e_{k-1}}$ preserves the degree of a symm. function and scales the **Macdonald basis** $\widetilde{H}_\mu$.

Rational Shuffle Thm [M]

$$E_{kn,km}\cdot 1 = (-1)^{k(m+1)}\sum_{P\in \mathrm{WLD}_{kn,km}} q^{\mathrm{dinv}(P)}t^{\mathrm{area}(P)}x_P. \tag{1}$$

The Rational Shuffle Thm was conjectured by Bergeron–Garsia–Levin–Xin and proven by Mellit. Note that when $m=1$ the sign of the RHS is positive, and the slope is an integer.

Example: For $n=3$, $m=1$ and $k=3$,

$P=$ 

$\mathrm{dinv}(P)=2$

$\mathrm{area}(P)=1$

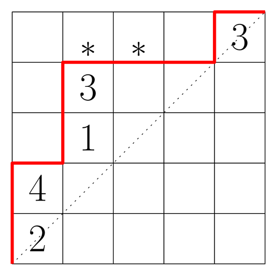
$x_P = x_1x_2x_3^2x_4x_6x_7^2x_9$

Delta Theorem (Fall version) [DM,BHMPS]

$$\Delta'_{e_{k-1}}e_n = \sum_{P\in \mathrm{WLD}^{\mathrm{fall}}_{n,k}} q^{\mathrm{dinv}(P)}t^{\mathrm{area}^-(P)}x_P. \tag{2}$$

The Delta Thm was conjectured by Haglund–Remmel–Wilson and proven by D'Adderio–Mellit and independently by Blasiak–Haiman–Morse–Pun–Seelinger.

Example: For $n=5$ and $k=3$,

$P=$ 

$\mathrm{dinv}(P)=2$

$\mathrm{area}^-(P)=1$

$x_P = x_1x_2x_3^2x_4$

Skewing identity

The Rational Shuffle Thm and Delta Thm are directly linked by the following skewing identity.

Main Theorem 1

Let $\mu=(k-1,\dots,k-1)=(k-1)^{n-k}$, a rectangular partition with $n-k$ parts. Then

$$\Delta'_{e_{k-1}}e_n = s_\mu^\perp(E_{k(n-k+1),k}\cdot 1). \tag{3}$$

Here, $s_\mu^\perp$ is the operator adjoint to multiplication by the Schur function s_μ.

Example: $(n=3,k=2)$

$$(1+q+t)s_{21}+(q+t+q^2+qt+t^2)s_{111} = s_{(1)}^\perp(s_{22}+(q+t)s_{211}+(q^2+qt+t^2)s_{1111}).$$

Algebraic proof of skewing identity

Let $H^k_{q,t}$ be the "raising" operator on symmetric rational functions in finitely many variables $\varphi(x_1,\dots,x_k)$ given by

$$H^k_{q,t}(\varphi) = \sum_{w\in S_k} w\left(\frac{\varphi(x)\prod_{i<j}(1-qtx_i/x_j)}{\prod_{i<j}(1-x_j/x_i)(1-qx_i/x_j)(1-tx_i/x_j)}\right).$$

It follows from work of Negut that

$$\omega(E_{k(n-k+1),k}\cdot 1)(x_1,\dots,x_k) = H^k_{q,t}\left(\frac{x_1^{n-k+1}\cdots x_k^{n-k+1}}{\prod(1-qtx_i/x_{i+1})}\right)_{\mathrm{pol}}. \tag{4}$$

Here f_{pol} is the result of expanding f into rational characters of GL_k (sums of $(x_1\cdots x_k)^{-m}s_\lambda$), then truncating the sum to only polynomial characters (sums of s_λ with no denominator).

Similarly, it follows from BHMPS that

$$\omega\Delta'_{e_{k-1}}e_n(x_1,\dots,x_k) = H^k_{q,t}\left(\frac{x_1\cdots x_k h_{n-k}(x_1,\dots,x_k)}{\prod(1-qtx_i/x_{i+1})}\right)_{\mathrm{pol}}. \tag{5}$$

There is no loss of information by truncating to k variables since all symm. functions involved can be recovered from truncating to $x_1,\dots,x_k$.

We show that $s^\perp_{\mu^t}$ applied to the RHS of (4) gives the RHS of (5). This proves (3).

Combinatorial proof of skewing identity

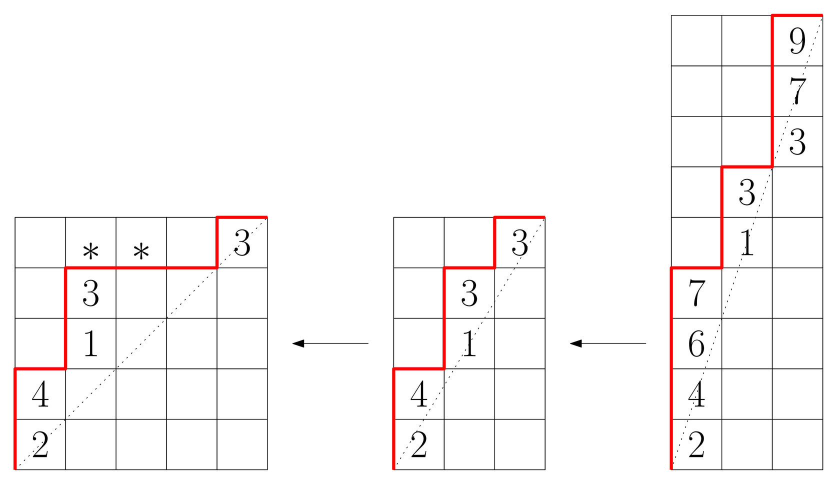
We give a combinatorial proof that when $m=1$, applying $s_\mu^\perp$ to the RHS of (1) is equal to the RHS of (2) (with n replaced by $n-k+1$ in (1)). Abbreviate

$$\mathrm{CombShuff} := (\text{RHS of (1) at } m=1) \quad = \sum_{P\in \mathrm{WLD}_{k(n-k+1),k}} q^{\mathrm{dinv}(P)}t^{\mathrm{area}(P)}x_P,$$

$$\mathrm{Comb}\Delta := (\text{RHS of (2)}) \quad = \sum_{P\in \mathrm{WLD}^{\mathrm{fall}}_{n,k}} q^{\mathrm{dinv}(P)}t^{\mathrm{area}^-(P)}x_P.$$

We show $\mathrm{Comb}\Delta = s^\perp_{(k-1)^{n-k}}(\mathrm{CombShuff})$ in steps as follows:

1. The identity is equivalent to $\langle h_\nu, \mathrm{Comb}\Delta\rangle = \langle h_\nu s_{(k-1)^{n-k}}, \mathrm{CombShuff}\rangle$.
2. Use Jacobi–Trudy identity to expand $s_{(k-1)^{n-k}} = \det(h_{k-1+i-j})_{i,j} = \sum_{\alpha \text{ allowable}}(-1)^{\mathrm{sgn}(\alpha)}h_\alpha$, so step 1 is equivalent to
$$\langle h_\nu, \mathrm{Comb}\Delta\rangle = \sum_{\alpha \text{ allowable}}(-1)^{\mathrm{sgn}(\alpha)}\langle h_{\nu,\alpha}, \mathrm{CombShuff}\rangle$$
$$\sum_{\substack{P\in \mathrm{WLD}^{\mathrm{fall}}_{n,k}\\ \mathrm{type}(P)=\nu}} q^{\mathrm{dinv}(P)}t^{\mathrm{area}^-(P)} = \sum_{\alpha \text{ allowable}}(-1)^{\mathrm{sgn}(\alpha)}\sum_{\substack{P\in \mathrm{WLD}_{k(n-k+1),k}\\ \mathrm{type}(P)=(\nu,\alpha)}} q^{\mathrm{dinv}(P)}t^{\mathrm{area}(P)}$$
3. We construct a sign-reversing involution on the objects on the RHS which changes the sign of α. We then give a weight-preserving bijection between the fixed points and the terms of the LHS.
For example, when $n=5$, $k=3$, $\nu=(1,1,2,1)$, $\alpha=(0,1,2,0,1)$, the fixed-point of type (ν,α) on the right becomes the fall-starred labeling of type ν on the left after deleting the α labels (meaning $5,6,7,8,9$) and expanding the columns:



Geometric interpretation

We explain how both the Rational Shuffle Thm and Delta Thm can be interpreted geometrically in terms of affine Springer fibers. We then show how a **geometric skewing identity** relates the two, thus giving a geometric interpretation of the skewing identity (3).

Affine flag varieties

Let $\mathcal{O}=\mathbb{C}[\![\epsilon]\!]$ and $K=\mathbb{C}(\!(\epsilon)\!)$.

A **lattice** is a $\mathcal{O}$-submodule of K^n of rank n, such as $\Lambda=\mathcal{O}\{e_1+\epsilon^{-1}e_3, \epsilon e_2, e_3, \epsilon^2 e_4, \epsilon^{-1}e_5\}\subset K^5$.

A **complete flag of lattices** $\Lambda_\bullet$ is $\Lambda_0\supset\Lambda_1\supset\cdots\supset\Lambda_{n-1}\supset\Lambda_n=\epsilon\Lambda_0$ with $\dim_{\mathbb{C}}(\Lambda_i/\Lambda_{i+1})=1$.

The **affine flag variety** is $\widetilde{Fl}_n := \{\Lambda_\bullet \text{ complete flags in } K^n\}$.

$\widetilde{Fl}_n$ has infinitely many connected components indexed by $\pi_1(GL_n)\cong\mathbb{Z}$.

Define the **positive normalized part** of $\widetilde{Fl}_n$ to be

$$\widetilde{Fl}_n^{+,0} := \{\Lambda_\bullet\in\widetilde{Fl}_n \mid \Lambda_0\subset\mathcal{O}^n,\ \Lambda_0\not\subset\epsilon\mathcal{O}\oplus\mathcal{O}^{n-1}\}.$$

Affine Springer fibers

Given $\gamma\in\mathfrak{gl}_nK$ such that $\lim_{m\to\infty}\gamma^m=0$, its **affine Springer fiber** is

$$\widetilde{Fl}_\gamma := \{\Lambda_\bullet\in\widetilde{Fl}_n \mid \gamma\Lambda_i\subset\Lambda_i\}.$$

By work of Lusztig, there is an action of S_n (restricted from the affine symmetric group) on the Borel–Moore homology $H_*(\widetilde{Fl}_n)$.

Take $\gamma=\begin{pmatrix}0 & \epsilon^2\\ \epsilon I_{n-1} & 0\end{pmatrix}$.

Theorem (Hikita) For the particular γ chosen above,

$$\mathrm{grFrob}(H_*(\widetilde{Fl}_\gamma);q,t) = \omega\circ\mathrm{rev}_q\nabla e_n.$$

Here, ∇e_n is the $n=k$ case of both (1) and (2), $\nabla e_n = \Delta'_{e_{n-1}}e_n = E_{n,n}\cdot 1$.

(Technically, Hikita works in $SL_n(K)/SL_n(\mathcal{O})$ not $GL_n(K)/GL_n(\mathcal{O})$)

Delta-Affine Springer fibers

Let $\gamma=\begin{pmatrix}0 & 0 & 0\\ 0 & \epsilon I_{k-1} & 0\\ I_{k(n-k)} & 0 & 0\end{pmatrix}$.

Main Theorem 2 We have

$$\mathrm{grFrob}\left(H_*\left(\widetilde{Fl}_\gamma^{+,0}\right);q,t\right) = \omega\circ\mathrm{rev}_q(E_{k(n-k+1),k}\cdot 1).$$

Given a composition $\alpha=(\alpha_1,\dots,\alpha_l)$, let $\widetilde{Fl}_\alpha$ be the corresponding **partial affine flag variety** of flags such that $\dim_{\mathbb{C}}(\Lambda_i/\Lambda_{i+1})=\alpha_i$.

Given a tuple of partitions $\vec{\lambda}=(\lambda^1,\dots,\lambda^l)$ with $\lambda^i\vdash\alpha_i$, define

$$BM_{\vec{\lambda},\gamma} := \{\Lambda_\bullet\in\widetilde{Fl}_\alpha \mid \gamma\Lambda_i\subset\Lambda_i,\ \mathrm{JT}(\gamma|_{\Lambda_i/\Lambda_{i+1}})\leqslant\lambda^i\}.$$

Main Theorem 3 For $\vec{\lambda}=((n-k)^{k-1},(1),(1),\dots,(1))$, we have

$$\mathrm{grFrob}\left(H_*\left(BM^{+,0}_{\vec{\lambda},\gamma}\right);q,t\right) = \omega\circ\mathrm{rev}_q(\Delta'_{e_{k-1}}e_n).$$

The proof of Theorem 3 relies on combining Theorems 1 and 2 with a **geometric skewing identity** which we prove using work of Borho and MacPherson.

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