Monotonicity for generalized binomial coefficients and Jack positivity

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Abstract

We study generalized binomial coefficients associated with interpolation polynomials. We prove positivity and monotonicity properties and derive inequalities for Jack, Schur, monomial, and elementary symmetric polynomials, generalizing several classical results.

Motivations and Preliminaries

The following **Newton's binomial formula** is well-known:

$$(x+1)^n = \sum_m \binom{n}{m} x^m. \tag{1}$$

The **binomial coefficient** $\binom{n}{m} = \frac{n(n-1)\cdots(n-m+1)}{m!}$ satisfies many simple properties:

- (Polynomiality) $\binom{n}{m}$ is a polynomial;
- (Positivity) $\binom{n}{m} > 0$ if $n \geqslant m$;
- (Vanishing) $\binom{n}{m} = 0$ unless $n \geqslant m$;
- (Monotonicity) $\binom{n}{k} \geqslant \binom{m}{k}$ if $n \geqslant m$.

One natural generalization is to consider **symmetric polynomials** in n variables, indexed by partitions of length at most n. Such a **partition** is an n-tuple $\lambda = (\lambda_1, \ldots, \lambda_n)$ such that $\lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant 0$. The monomial m_λ, elementary e_λ, Schur s_λ and power-sum p_λ are the most basic examples of symmetric polynomials. The **Jack polynomials** form a one-parameter family of symmetric polynomials depending on a parameter τ (where $\tau = 1/\alpha$); they specialize to the monomial, Schur, and transposed elementary when $\tau = 0$, 1, and ∞, respectively.

Okounkov–Olshanski generalized the binomial formula to Jack polynomials:

$$\frac{P_\lambda(x+\mathbf{1};\tau)}{P_\lambda(\mathbf{1};\tau)} = \sum_\mu \binom{\lambda}{\mu}_\tau \frac{P_\mu(x;\tau)}{P_\mu(\mathbf{1};\tau)}, \tag{2}$$

where $x = (x_1, \ldots, x_n)$, $\mathbf{1} = (1, \ldots, 1)$, P_λ is the monic Jack polynomial, and $\binom{\lambda}{\mu}_\tau$ is the **generalized binomial coefficient**. Okounkov–Olshanski showed that the generalized binomial coefficients are evaluations of **interpolation Jack polynomials** (also known as shifted Jack polynomials), first studied by Knop–Sahi.

The *unital* **interpolation polynomial** h_μ is the unique symmetric polynomial over $\mathbb{F} = \mathbb{Q}(\tau)$ satisfying the interpolation condition and the degree condition:

$$h_\mu(\overline{\lambda};\tau) = \delta_{\lambda\mu}, \quad |\lambda| \leqslant |\mu|; \qquad \deg h_\mu = |\mu|,$$

where $|\lambda| = \sum \lambda_i$ is the size of λ and the shifting is given by $\overline{\lambda}_i = \lambda_i + (n-i)\tau$.

For example, when $n = 2$, $\mu = (3,2)$, the *monic* interpolation Jack polynomial is

$$h^{\text{monic}}_{(3,2)}(x_1, x_2; \tau) = x_1 x_2 (x_1 - 1)(x_2 - 1)(x_1 + x_2 - \tau - 4).$$

One can easily verify that $h^{\text{monic}}_{(3,2)}(\overline{(3,2)}) \neq 0$ and $h^{\text{monic}}_{(3,2)}$ vanishes at $\overline{(2,2)} = (2+\tau, 2)$, $\overline{(m,0)} = (m+\tau, 0)$ and $\overline{(m-1,1)} = (m-1+\tau, 1)$, more points than required in the definition. In general, the generalized binomial coefficients satisfy the **extra vanishing property** and are defined by:

$$\binom{\lambda}{\mu}_\tau := h_\mu(\overline{\lambda};\tau) = 0 \quad \text{unless} \quad \lambda \supseteq \mu. \tag{3}$$

Here, we say λ **contains** μ, denoted by $\lambda \supseteq \mu$, if $\lambda_i \geqslant \mu_i$, $1 \leqslant i \leqslant n$.

When $\mu = (1)$ or $|\lambda| = |\mu| + 1$, the binomial coefficients can be computed by some combinatorial formulas.

Some Inequalities

Recall that for partitions λ and μ of length at most n, we say λ **weakly dominates** μ, if $\sum_{i=1}^r \lambda_i \geqslant \sum_{i=1}^r \mu_i$, for $1 \leqslant r \leqslant n$; if, in addition, $|\lambda| = |\mu|$, we say λ **dominates** μ.

In the work of Muirhead, Cuttler–Greene–Skandera and Sra, they showed that dominance can be characterized by the following inequalities: assume $|\lambda| = |\mu|$,

$$\lambda \text{ dominates } \mu \iff \frac{m_\lambda}{m_\lambda(\mathbf{1})} - \frac{m_\mu}{m_\mu(\mathbf{1})} \geqslant 0 \iff \frac{e'_\lambda}{e'_\lambda(\mathbf{1})} - \frac{e'_\mu}{e'_\mu(\mathbf{1})} \geqslant 0$$
$$\iff \frac{p_\lambda}{p_\lambda(\mathbf{1})} - \frac{p_\mu}{p_\mu(\mathbf{1})} \geqslant 0 \iff \frac{s_\lambda}{s_\lambda(\mathbf{1})} - \frac{s_\mu}{s_\mu(\mathbf{1})} \geqslant 0.$$

Here, $f \geqslant 0$ means that $f(x) \geqslant 0$ for $x \in [0,\infty)^n$.

Khare–Tao showed a similar characterization for weak dominance:

$$\lambda \text{ weakly dominates } \mu \iff \frac{s_\lambda(x+\mathbf{1})}{s_\lambda(\mathbf{1})} - \frac{s_\mu(x+\mathbf{1})}{s_\mu(\mathbf{1})} \geqslant 0, \quad \forall x \in [0,\infty)^n.$$

Our Work

Theorem 1 (Positivity and Monotonicity)

For $\tau > 0$, we have

- *(Positivity) $\binom{\lambda}{\mu} > 0$ if and only if $\lambda \supseteq \mu$;*
- *(Monotonicity) $\binom{\lambda}{\nu} \geqslant \binom{\mu}{\nu}$ if $\lambda \supseteq \mu$.*

As an application of the monotonicity, we have the following characterization:

Theorem 2 (Characterization of Containment)

We have λ contains μ if and only if the following expressions are ***expansion positive****:*

- $\dfrac{s_\lambda(x+\mathbf{1})}{s_\lambda(\mathbf{1})} - \dfrac{s_\mu(x+\mathbf{1})}{s_\mu(\mathbf{1})}$
- $\dfrac{e_\lambda(x+\mathbf{1})}{e_\lambda(\mathbf{1})} - \dfrac{e_\mu(x+\mathbf{1})}{e_\mu(\mathbf{1})}$
- $\dfrac{m_\lambda(x+\mathbf{1})}{m_\lambda(\mathbf{1})} - \dfrac{m_\mu(x+\mathbf{1})}{m_\mu(\mathbf{1})}$
- $\dfrac{P_\lambda(x+\mathbf{1};\tau)}{P_\lambda(\mathbf{1};\tau)} - \dfrac{P_\mu(x+\mathbf{1};\tau)}{P_\mu(\mathbf{1};\tau)}$

where the Jack positivity is over $\mathbb{F}_{\geqslant 0} := \{f(\tau) \in \mathbb{Q}(\tau) \mid f(\tau_0) \geqslant 0, \forall \tau_0 \in [0,\infty]\}$.

For example, write $S_\lambda(x) = s_\lambda(x)/s_\lambda(\mathbf{1})$ and $\widetilde{S}_\lambda(x) = S_\lambda(x+\mathbf{1})$, and similarly for M and $\widetilde{M}$, E and $\widetilde{E}$, $\Omega = P_\lambda(x)/P_\lambda(\mathbf{1})$ and $\widetilde{\Omega}$, then for $\lambda = (3,1)$ and $\mu = (2)$

$$\widetilde{S}_{(3,1)} - \widetilde{S}_{(2)} = S_{(3,1)} + \tfrac{4}{3} S_{(3)} + \tfrac{8}{3} S_{(2,1)} + 3S_{(2)} + 2S_{(1,1)} + 2S_{(1)};$$
$$\widetilde{M}_{(3,1)} - \widetilde{M}_{(2)} = M_{(3,1)} + M_{(3)} + 3M_{(2,1)} + 2M_{(2)} + 3M_{(1,1)} + 2M_{(1)};$$
$$\widetilde{E}_{(2,1,1)} - \widetilde{E}_{(1,1)} = E_{(2,1,1)} + 2E_{(1,1,1)} + 2E_{(2,1)} + 4E_{(1,1)} + E_{(2)} + 2E_{(1)};$$
$$\widetilde{\Omega}_{(3,1)} - \widetilde{\Omega}_{(2)} = \Omega_{(3,1)} + \tfrac{2\tau+2}{\tau+2}\Omega_{(3)} + \tfrac{2\tau+6}{\tau+2}\Omega_{(2,1)} + \tfrac{4\tau+2}{\tau+1}\Omega_{(2)} + \tfrac{\tau+3}{\tau+1}\Omega_{(1,1)} + 2\Omega_{(1)}.$$

We end with two conjectures generalizing the inequalities of Cuttler–Greene–Skandera and Khare–Tao to Jack polynomials:

Conjecture

Let λ and μ be partitions of length at most n, P_λ be the monic Jack polynomial, and let $\mathbb{F}^{\mathbb{R}}_{\geqslant 0} := \{ f(\tau) \in \mathbb{R}(\tau) \mid f(\tau_0) \geqslant 0, \forall \tau_0 \in [0,\infty] \}$.

- *(CGS Conjecture for Jack polynomials) Assume $|\lambda| = |\mu|$. λ dominates μ if and only if*
$$\frac{P_\lambda(x;\tau)}{P_\lambda(\mathbf{1};\tau)} - \frac{P_\mu(x;\tau)}{P_\mu(\mathbf{1};\tau)} \in \mathbb{F}^{\mathbb{R}}_{\geqslant 0}, \quad \forall x \in [0,\infty)^n. \tag{4}$$
- *(KT Conjecture for Jack polynomials) λ weakly dominates μ if and only if*
$$\frac{P_\lambda(x+\mathbf{1};\tau)}{P_\lambda(\mathbf{1};\tau)} - \frac{P_\mu(x+\mathbf{1};\tau)}{P_\mu(\mathbf{1};\tau)} \in \mathbb{F}^{\mathbb{R}}_{\geqslant 0}, \quad \forall x \in [0,\infty)^n. \tag{5}$$

Note that the "if" direction of the two conjectures can be easily proved by some degree consideration; also the CGS conjecture, together with Theorem 2, implies the KT conjecture. In other words, the only missing part is

$$\lambda \text{ dominates } \mu \implies \text{Eq. (4).}$$

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