# Crystal skeletons and their axioms

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## Introduction

One important problem in representation theory and algebraic combinatorics is to deduce the Schur function expansion of a symmetric function whose expansion in terms of Gessel's fundamental quasisymmetric functions is known. Toward this goal, crystal skeletons were introduced in [3] as a tool for interpolating between crystal graphs and their quasi-crystal components. In [2], we combinatorially classify and axiomatize the structure of crystal skeletons.

## Crystals, quasi-crystals, and crystal skeletons

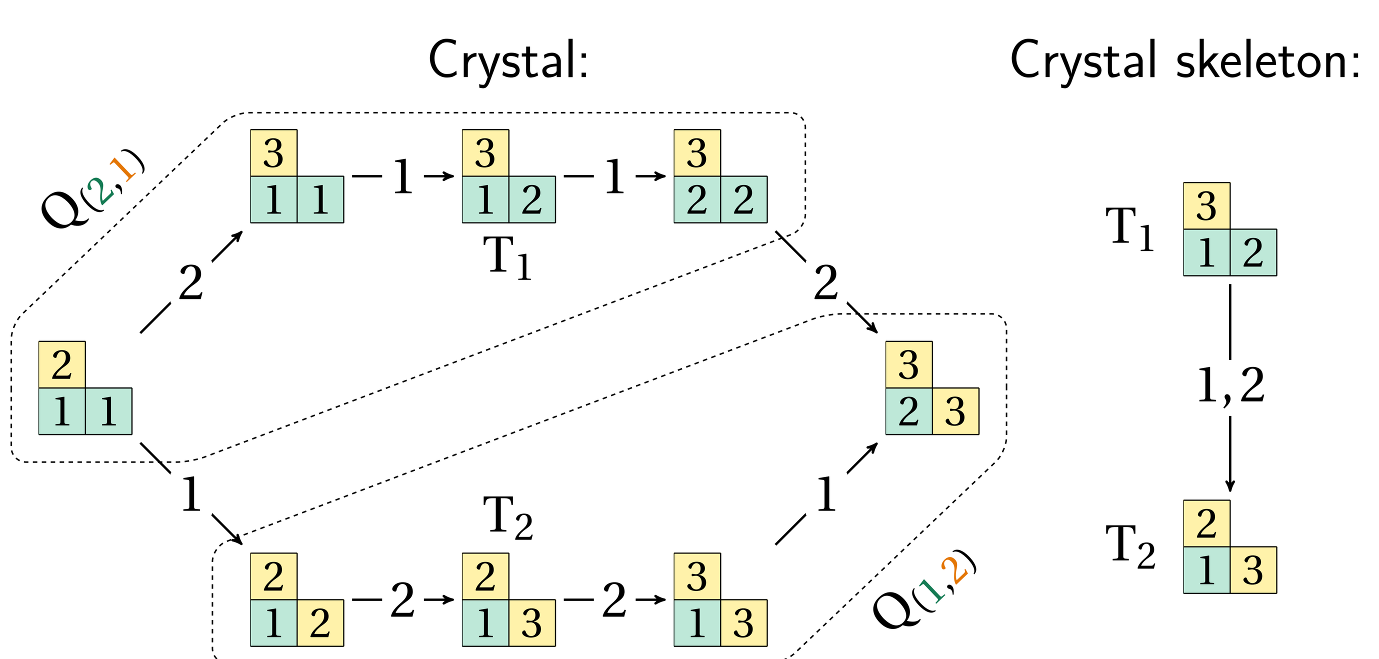
For a partition $\lambda \vdash n$, the type-$A_{n-1}$ **crystal graph** $B(\lambda)_n$ has vertices indexed by semistandard Young tableaux $S \in SSYT(\lambda)$ (with entries from $[n] = \{1, \ldots, n\}$), and directed edges labelled by integers $[n-1]$ corresponding to lowering operators.

Given a standard tableau T, the **quasi-crystal** $Q_T$ is the subgraph induced by those $S \in SSYT(\lambda)$ that standardize to T. Just as the character of $B(\lambda)$ is the Schur function $s_\lambda$, the character for the quasi-crystal $Q_T$ is Gessel's fundamental quasisymmetric function $F_{Des(T)}$.

The **crystal skeleton** $CS(\lambda)$ was initially defined by contracting each quasi-crystal component of $B(\lambda)_n$, resulting in a directed graph with vertices indexed by standard Young tableaux.

[*Note:* In this project, we relabel the edges of $CS(\lambda)$ using data from standard tableaux directly.]

Figure 1. **Left:** The crystal $B(2,1)_3$ with two quasi-crystal components outlined and standard tableaux indicated by $T_1$ and $T_2$. **Right:** the contraction of each quasi-crystal.




## Acknowledgements

This research was partially supported by ANR grant ANR-19-CE48-0011; NSF grants DMS-2303060, DMS-2054482, and DMS-2053350; and Simons Foundation grant MPS-TSM-00007191; Reed College; and the Banff International Research Station.

## Combinatorial data

Given $S \in SSYT(\lambda)$ ,

- row(S) is the word given by reading entries left-to-right, top-to-bottom (in French notation);
- Des(S) is the (right) descent composition of row(S).

For a standard tableau T, a **Dyck pattern interval** is any $[i, i+2m] \subseteq [n]$ for which the RSK insertion tableaux of $row(T)|_{[i,i+2m]}$ and $row(T)|_{[i,i+m]}$ have shape $(m+1, m)$ and $(m+1)$, respectively.

**Example.**

$$T = \begin{array}{ccc} 5 & & \\ 2 & 6 & \\ 1 & 3 & 4 \end{array} \qquad row(T) = 5\,2\,6\,1\,3\,4 \qquad Des(T) = (1, 3, 2)$$

The interval I = [2,6] is a Dyck pattern interval in row(T) since

$$row(T)|_{[2,6]} = 5\,2\,6\,3\,4 \xrightarrow{\text{RSK insertion}} \begin{array}{ccc} 5 & 6 & \\ 2 & 3 & 4 \end{array}$$

$$row(T)|_{[2,4]} = 2\,3\,4 \xrightarrow{\text{RSK insertion}} \begin{array}{ccc} 2 & 3 & 4 \end{array}$$

## Edges

**Thm.** The edges $T \to T'$ in $CS(\lambda)$ are in bijection with Dyck pattern intervals $I = [i, i+2m]$ in T. Moreover, there is a cycle of the form $\sigma = (j+m, j+m-1, \ldots, j-1, j)$ for which $\sigma \cdot T = T'$. See Figure 2.

**Cor.** The **dual equivalence graph** [1] of shape $\lambda$ is the subgraph of $CS(\lambda)$ induced by edges with intervals of length 3.

## Selected examples of notable structure in $CS(\lambda)$

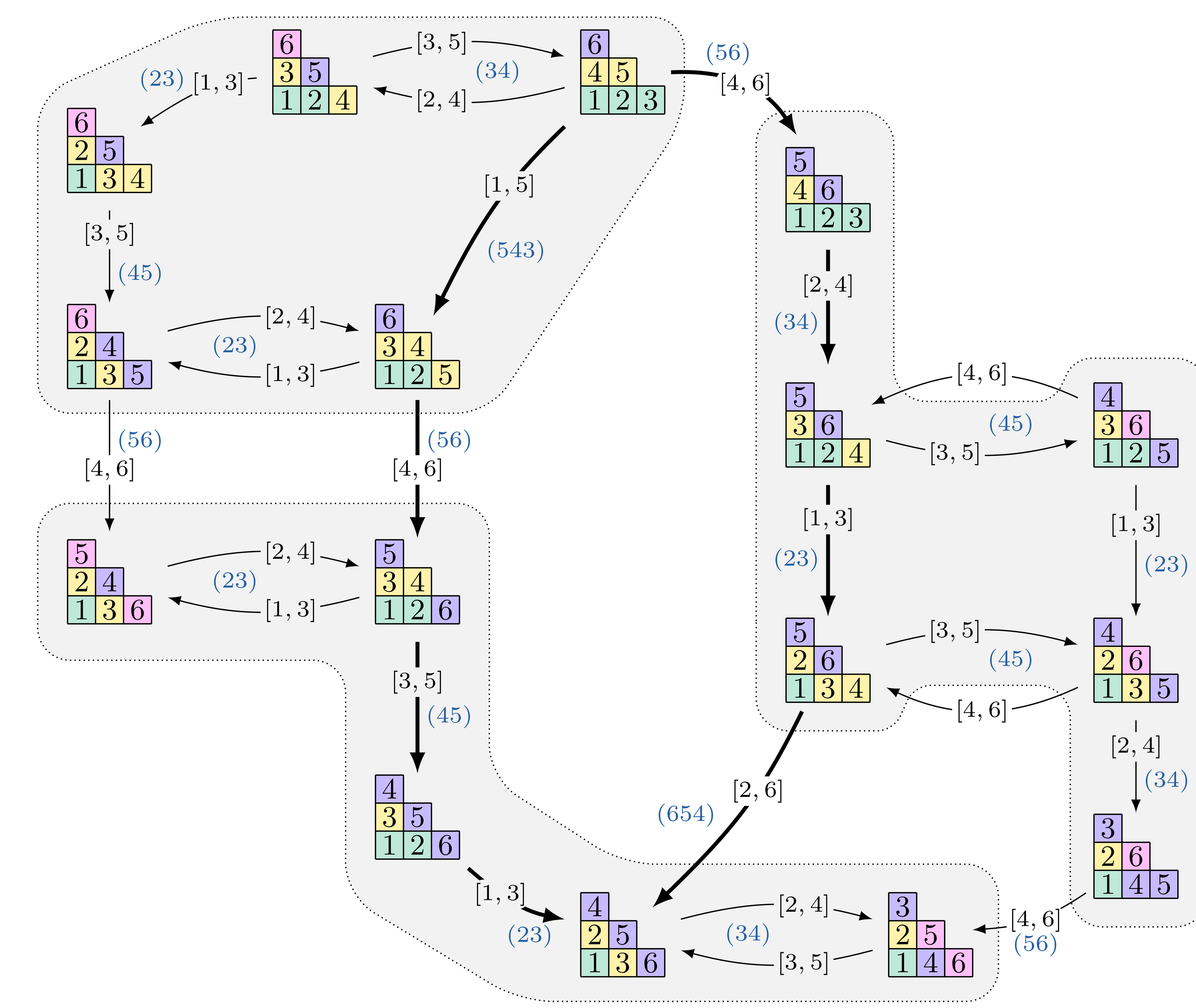
**Lusztig involution** $\eta : B(\lambda) \to B(\lambda)$ is defined in type A via evacuation: For $S \in SSYT(\lambda)$ and $row(S) = w_1 \ldots w_\ell$, define evac(T) as the RSK insertion tableau of the word $row(S)^\# = (n+1-w_\ell)\ldots(n+1-w_1)$. This induces an involution on $CS(\lambda)$ which sends Des(T) to its reverse, and

$$[i, i+2m] \mapsto [n+1-(i+2m), n+1-i].$$

**Self-similarity and branching:** For $T \in SYT(\lambda)$ and an interval $[a,b] \subseteq [n]$, let $T_{[a,b]}$ to be the skew tableau given by restricting to boxes filled by $[a,b]$. If $\mu$ is the shape of the jeu de taquin straightening of $T_{[a,b]}$, then $CS(\mu)$ is a natural subgraph of $CS(\lambda)$. It follows that deleting those edges in $CS(\lambda)$ with intervals containing $n$ results in a graph $G_{[1,n-1]}$ whose connected components are crystal skeletons $CS(\mu)$ with $\mu = \lambda - \square$.

**Top subcrystal:** The subgraph induced by tableaux with descent composition of length $\ell = \ell(\lambda)$ is isomorphic to the crystal $B(\lambda)_\ell$.

Figure 2. $CS(3,2,1)$ with edges labeled by the intervals and cycles. Thick arrows highlight the top subcrystal $B(3,2,1)_3$, and the outlined components are the connected components in the restriction $G_{[1,5]}$.



## Axioms

In [2], we give three axiomatizations of crystal skeletons, analogous to Stembridge's axioms for crystal graphs. Each starts with:

- a directed graph with vertices labeled by integer compositions of $n$ and edges labeled by odd-length intervals $I \subseteq [n]$;
- compatibility between incident compositions and intervals;
- existence conditions on edges.

Then, what we call $GL_n$ axioms and $S_n$ axioms specify local and global properties such as "fan" shaped subgraphs, Lusztig involution, top subcrystals, branching, and/or connectivity. Finally, local axioms instead condition on commutation relations and string lengths.

## References

[1] S. H. Assaf. "Dual equivalence graphs I: A new paradigm for Schur positivity". *Forum Math. Sigma* **3** (2015), Paper No. e12, 33.

[2] S. Brauner, S. Corteel, Z. Daugherty, and A. Schilling. "Crystal skeletons: Combinatorics and axioms". preprint, arXiv:2503.14782. 2025

[3] F. Maas-Gariëpy. "Quasicrystal structure of fundamental quasisymmetric functions, and skeleton of crystals. preprint, arXiv:2302.07694. 2023.