Most q-matroids are not representable

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Motivation and Intuition

In combinatorics the q-analogue of a problem can be interpreted as what happens if we generalize from finite sets to finite dimensional vector spaces. The "q" is often seen as the size of a finite field.

Intuitively we can view q-matroids as q-analogues of matroids.

The original motivation for defining q-matroids comes from algebraic coding theory. In particular, the so called **representable** q-matroids arise from vector-rank-metric codes. Hence by studying those q-matroids we can gain information about the associated codes [5].

Matroids

Matroids: basics [7, Chapter 1]

Matroid: Pair $M = (E, r)$ of a finite set E and function $r : \mathcal{P}(E) \to \mathbb{Z}$ satisfying the *rank-axioms*: for all $A, B \subseteq E$

1. $0 \leq r(A) \leq |A|$.
2. if $A \subseteq B$, then $r(A) \leq r(B)$.
3. $r(A \cap B) + r(A \cup B) \leq r(A) + r(B)$ (**semimodularity**).

- **Rank of M:** the value $r(M) := r(E)$.
- **Basis:** subset $B \subseteq E$ s.t. $r(B) = |B| = r(E)$.
- **Circuit:** $C \subseteq E$ s.t. $r(C) = |C| - 1$ and $r(A) = |A|$ for all $A \subsetneq C$.
- A matroid M is **paving** if every circuit C of M satisfies $|C| \geq r(M)$.

Matroids: motivating example

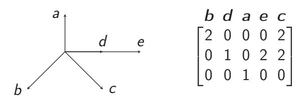

Figure 1: Config. of 5 vectors in $\mathbb{R}^3$.

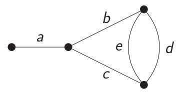

Figure 2: Graph with 5 labled edges.

Matroids: representability [7, Chapter 6]

- Let E be the set of column labels of a full-rank $(k \times n)$-matrix A over a field $\mathbb{F}$. Let $\mathbb{B}_A$ be the collection of k-subsets $X = \{i_1, \ldots, i_k\} \subseteq E$ s.t. $\det(A[i_1, \ldots, i_k]) \neq 0$.
- **Vector matroid of A:** $M[A] := (E, \mathbb{B}_A)$.
- A rank-k matroid M on n-elements is $\mathbb{F}$-**representable**, if there exist a full-rank $(k \times n)$-matrix A over $\mathbb{F}$ s.t $M = M[A]$.

Nelson '18: representability in the limit case [6]

Asymptotically almost all matroids are non-representable.

Vámos matroid

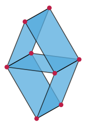

Figure 3: Geometric representation.

- Paving matroid of rank 4 over 8 elements.
- The *smallest* non-representable matroid.

q-Matroids

q-Matroids: basics [5]

q-Matroid: Pair $\mathcal{M} = (E, \rho)$ of a finite dim. vector space E over field $\mathbb{F}$ and a function $\rho : \mathcal{L}(E) \to \mathbb{Z}$ satisfying the *q-rank-axioms*: for all $X, Y \leq E$

1. $0 \leq \rho(X) \leq \dim(X)$.
2. if $X \leq Y$, then $\rho(X) \leq \rho(Y)$.
3. $\rho(X \cap Y) + \rho(X + Y) \leq \rho(X) + \rho(Y)$ (**semimodularity**).

- **Rank of $\mathcal{M}$:** the value $\rho(\mathcal{M}) := \rho(E)$.
- **Basis:** subspace $B \leq E$ s.t. $\rho(B) = \dim(B) = \rho(E)$.
- **Circuit:** $C \leq E$ s.t. $\rho(C) = \dim(C) - 1$ and $\rho(A) = \dim(A)$ for all $A \lneq C$.
- A q-matroid $\mathcal{M}$ is **paving** if every circuit C of $\mathcal{M}$ satisfies $\dim(C) \geq \rho(\mathcal{M})$.

q-Matroids: representability [5]

- Let $E = \mathbb{F}_q^n$ (q prime) and $\mathcal{C} \leq \mathbb{F}_{q^m}^n$ a k-dim. $\mathbb{F}_{q^m}$-space for some $m \geq 1$. Let $G \in \mathbb{F}_{q^m}^{k \times n}$ be the generator matrix, i.e. $\mathrm{RS}_{\mathbb{F}_{q^m}}(G) = \mathcal{C}$.
- **q-matroid associated to $\mathcal{C}$:** $\mathcal{M}_{\mathcal{C}} := (E, \rho_{\mathcal{C}})$, where:
$$\rho_{\mathcal{C}}(V) = \mathrm{rank}_{\mathbb{F}_{q^m}}(GA^T), \quad \text{for } V = \mathrm{RS}_{\mathbb{F}_q}(A).$$
- $\mathcal{C}$ is a $\mathbb{F}_{q^m}$-space of the metric space $(\mathbb{F}_{q^m}^n, \mathrm{d}_{\mathrm{rk}})$, where: $\forall u, v \in \mathbb{F}_{q^m}^n$
$$\mathrm{d}_{\mathrm{rk}}(u, v) := \mathrm{rk}(u - v) \quad \text{with} \quad \mathrm{rk}(v) := \dim_{\mathbb{F}_q}\langle v_1, \ldots, v_n \rangle.$$
$\mathcal{C}$ is called $[n, k]_{q^m/q}$-**code**.
- A k-rank q-matroid $\mathcal{M} = (E, \rho)$ is **representable**, if there exists $m \in \mathbb{Z}_{>0}$ and an $[n, k]_{q^m/q}$-code $\mathcal{C}$ s.t. $\mathcal{M} = \mathcal{M}_{\mathcal{C}}$.

q-Matroids: example [1]

The q-matroids $\mathcal{M}, \mathcal{M}^*$ in Fig. (4)-(5) are both representable, via the following matrices:

$$\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}_{\mathbb{F}_{2^1}}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}_{\mathbb{F}_{2^1}}.$$

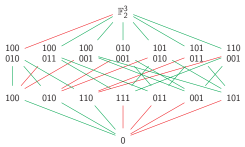

Figure 4: Bicolored subspace lattice of $\mathcal{M}$.

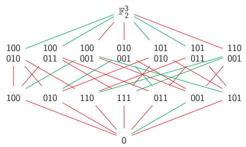

Figure 5: Bicolored subspace lattice of $\mathcal{M}^*$.

Relations to algebra and coding theory

Coding theory: constant dimension codes [4]

- Let $(\mathcal{L}(\mathbb{F}_q^n), d_S)$ be a metric space, with **subspace distance** d_S defined as: for all $V, W \in \mathcal{L}(\mathbb{F}_q^n)$
$$d_S(V, W) := \dim(V) + \dim(W) - 2\dim(V \cap W).$$
- **k-constant dimension code (k-CDC):** Non-empty subset $\mathcal{C} \subseteq \mathcal{L}(\mathbb{F}_q^n)$ s.t. all elements are of equal dimension k.
- The maximal cardinality of a k-CDC has an exponential lower bound.

Relating CDC's and paving q-matroids

[2, Lemma 3.3]: Let $\mathcal{S} \subseteq \mathcal{L}(\mathbb{F}_q^n)$ be a k-CDC, s.t. $d_S(V, W) \geq 4$ for all $V, W \in \mathcal{S}$. Then $\mathcal{S}$ corresponds to a paving q-matroid of rank k.

- Also every subset of $\mathcal{S}$ gives rise to a paving q-matroid. Therefore we have $2^{|\mathcal{S}|}$ many of them.

Algebra: zero patterns [8]

- Let $\mathbb{K}$ be field and $a \in \mathbb{K}$, define
$$\delta(a) := \begin{cases} 0 & \text{if } a = 0, \\ * & \text{otherwise.} \end{cases}$$
Let $\mathcal{F} = (f_1, \ldots, f_m)$ be a sequence of polynomials in $\mathbb{K}[x_1, \ldots, x_s]$. For $a \in \mathbb{K}^s$ we call $\delta(\mathcal{F}, a) := (\delta(f_1(a)), \ldots, \delta(f_m(a)))$ a **zero pattern of $\mathcal{F}$**.
- The number of zero patterns of $\mathcal{F}$ has an exponential upper bound depending on the degrees of the polynomials in $\mathcal{F}$.

Relating zero patterns and representable q-matroids

- Let $n \geq 1$ and $1 \leq k \leq n$. Consider the vector space $\mathbb{F}_q^n$ and fix a total ordering on the set of all k-dimensional subspaces of $\mathbb{F}_q^n$, i.e., $U_1, \ldots, U_{\binom{n}{k}_q}$.
- To each k-dimensional space $U_i \in \mathcal{L}(\mathbb{F}_q^n)_k$ we can associated a homogeneous polynomial f_{U_i} of degree k in some polynomial ring P over $\overline{\mathbb{F}_q}$ with kn-many variables. *(Details: [2, Definition 4.3])*
- Denote by $\mathcal{F}_{n,k} := (f_{U_i})_{1 \leq i \leq \binom{n}{k}_q}$ the sequence of the above polynomials.
- Let $\mathcal{M}$ be a q-matroid of rank k on $\mathbb{F}_q^n$ and $\mathcal{B}, \mathcal{NB}$ its collection of bases and non-bases, respectively.

[2, Lemma 4.4]: $\mathcal{M}$ is representable if and only if there exists a zero pattern of $\mathcal{F}_{n,k}$ for some $u \in \overline{\mathbb{F}_q}^{kn}$ of the form
$$\delta(\mathcal{F}_{n,k}, u) = (f_{U_i}(u))_{1 \leq i \leq \binom{n}{k}_q} = \begin{cases} 0 & \text{if } U_i \in \mathcal{NB}, \\ * & \text{if } U_i \in \mathcal{B}. \end{cases}$$

Our Result

Motivating questions

1. **Question:** Are all q-matroids representable? (Jurrius, Pellikaan '18, [5])

   Answer: No!

- Luerssen, Jany '22 ([3]): Method of translating non-representable matroids to the q-analogue setting (e.g. q-Vámos matroid).
- Ceria, Jurrius '22 ([1]): Smallest non-representable q-matroid (rank 2 on $\mathbb{F}_2^4$).

2. **Question:** Is there a q-analogue of Nelson's theorem?

   Answer: Yes!

Main result: asymptotics of representable q-matroids [2]

Theorem (D., Kühne '24)

Let $\mathcal{R}_q(n)$ be the number of representable q-matroids and $\mathcal{N}_q(n)$ be the number of all q-matroids on $\mathbb{F}_q^n$, respectively. Then $\lim_{n \to \infty} \frac{\mathcal{R}_q(n)}{\mathcal{N}_q(n)} = 0$.

In orther words: Asymptotically almost all q-matroids are non-representable.

Proof strategy

- Find a lower bound on $\mathcal{N}_q(n)$, which is **at least doubly exponentially**.
  - → Use the relation between paving q-matroids and CDC's, together with the lower bounds on their maximal sizes.
  - → **[2, Theorem 3.5]**
- Find an upper bound on $\mathcal{R}_q(n)$, which is **at most exponentially**.
  - → Use the relation between representable q-matroids and zero patterns, together with their upper bounds.
  - → **[2, Theorem 4.7]**

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