The Redei-Berge function in noncommuting variables

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Abstract

Recently, Stanley and Grinberg introduced a symmetric function associated with digraphs. They named it the Redei-Berge symmetric function, in honor of two mathematicians, whose results about the number of Hamiltonian paths in a digraph they managed to deduce in a new way - using the theory of symmetric functions. This function, however, does not satisfy the deletion-contraction property, which is a very powerful tool for proving various identities using induction.

We introduce an analogue of this function in noncommuting variables which does have such property. Furthermore, it specializes to the ordinary Redei-Berge function when the variables are allowed to commute. This modification allows us to further generalize properties that are already proved for the original function and to deduce many new ones.

Digraphs

A digraph X is a pair $X = (V, E)$, where:

- V is a finite set, called the set of vertices
- E is a collection $E \subseteq V \times V$, called the set of edges.

A V-listing is a list of all elements of V with no repetitions. The set of all $V-$listings is denoted as Σ_V.

For a V-listing $\sigma = (\sigma_1, \dots, \sigma_n) \in \Sigma_V$, the $X-$descent set is the set

$$X\mathrm{Des}(\sigma) = \{1 \le i \le n-1 | (\sigma_i, \sigma_{i+1}) \in E\}.$$

Example : If X is $1 \longrightarrow 2 \longrightarrow 3$, then

$X\mathrm{Des}(1,2,3) = \{1,2\}$ $X\mathrm{Des}(1,3,2) = \emptyset$ $X\mathrm{Des}(2,1,3) = \emptyset$

$X\mathrm{Des}(2,3,1) = \{1\}$ $X\mathrm{Des}(3,1,2) = \{2\}$ $X\mathrm{Des}(3,2,1) = \emptyset$.

Ordinary Redei-Berge function

The Redei-Berge symmetric function of a digraph $X = (V, E)$ is

$$U_X = \sum_{\sigma \in \Sigma_V} F_{X\mathrm{Des}(\sigma)},$$

where F_I's are the fundamental quasisymmetric functions.

Example : If X is $1 \longrightarrow 2 \longrightarrow 3$, then $U_X = 3F_\emptyset + F_{\{1\}} + F_{\{2\}} + F_{\{1,2\}}$.

Noncommutative symmetric functions

Let Π_n denote the lattice of set partitions of $[n] = \{1, 2, \dots, n\}$ ordered by refinement.

For $\pi \in \Pi_n$, the noncommutative monomial symmetric function, m_π is

$$m_\pi = \sum_{i_1, i_2, \dots, i_n} x_{i_1} x_{i_2} \cdots x_{i_n},$$

where the sum is over all sequences $i_1, i_2, \dots, i_n$ of positive integers such that $i_j = i_k$ if and only if j and k are in the same block of π.

Their span is called the algebra of noncommutative symmetric functions.

The noncommutative power sum symmetric function p_π is

$$p_\pi = \sum_{\pi \le \sigma} m_\sigma.$$

The Redei-Berge function in noncommuting variables

Let $X = (V, E)$ be a digraph. For $f : V \to \mathbb{P}$, a V-listing $\sigma = (\sigma_1, \dots, \sigma_n) \in \Sigma_V$ is called $(f, X)-$friendly if

$$f(\sigma_1) \le f(\sigma_2) \le \dots \le f(\sigma_n) \text{ and}$$
$$f(\sigma_j) < f(\sigma_{j+1}) \text{ for each } j \in [n-1] \text{ satisfying } (\sigma_j, \sigma_{j+1}) \in E.$$

Let $\Sigma_V(f, X)$ denote the set of all $(f, X)-$friendly $V-$listings.

For a digraph $X = (V, E)$ with vertices labeled $v_1, v_2, \dots, v_n$ in fixed order, its Redei-Berge function in noncommuting variables is

$$W_X = \sum_{f: V \to \mathbb{P}} \sum_{\sigma \in \Sigma_V(f,X)} x_{f(v_1)} x_{f(v_2)} \cdots x_{f(v_n)}.$$

Allowing the variables commute transforms W_X to U_X.

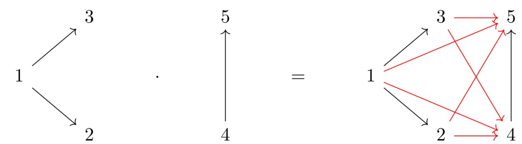
Example : If $D_n = ([n], \emptyset)$ is the discrete digraph with n vertices,

$$W_{D_n} = \sum_{\pi \in \Pi_n} \pi! m_\pi,$$

with $\pi! = 1!^{r_1} 2!^{r_2} \cdots n!^{r_n}$, where r_i is the number of blocks of π of size i.

Some properties of W_X

For $X = (V, E)$ and $Y = (V', E')$ we define the product $X \cdot Y$ as the digraph on the disjoint union $V \sqcup V'$ with the set of edges $E \cup E' \cup \{(u, v) \mid u \in V, v \in V'\}$.



The opposite digraph of a digraph $X = (V, E)$ is the digraph $X^{op} = (V, E')$, where $E' = \{(v, u) \mid (u, v) \in E\}$.

For any two labeled digraphs X and Y,

$$W_{X \cdot Y} = W_X \cdot W_Y \text{ and } W_X = W_{X^{op}}.$$

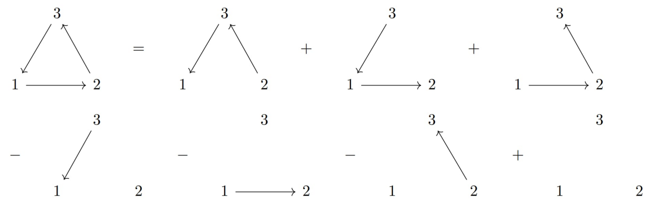
Decomposition techniques

If $X = (V, E)$ is a labeled digraph that is not a disjoint union of paths, then

$$W_X = \sum_{S \subseteq E, S \ne \emptyset} (-1)^{|S|-1} W_{X \setminus S}.$$

If $e_1, e_2, \dots, e_k$ is a list of edges that form a directed cycle in a digraph $X = (V, E)$, then

$$W_X = \sum_{\substack{S \subseteq \{e_1, e_2, \dots, e_k\} \\ S \ne \emptyset}} (-1)^{|S|-1} W_{X \setminus S}.$$



The deletion-contraction property

The deletion of an edge $e \in E$ from a digraph $X = (V, E)$ is the digraph $X \setminus e = (V, E \setminus \{e\})$.

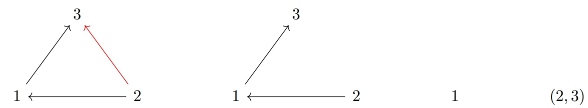
The contraction of an edge $e = (u, v) \in E$ from a digraph $X = (V, E)$ is the digraph $X/e = (V', E')$, where $V' = V \setminus \{u, v\} \cup \{e\}$ and E' contains all edges in E with vertices different from $u, v \in V$ and additionally for $w \ne u, v$ we have

- $(w, e) \in E'$ if and only if $(w, u) \in E$ and
- $(e, w) \in E'$ if and only if $(v, w) \in E$.

If $X = (V, E)$ is any digraph with vertices labeled $v_1, v_2, \dots, v_n$ such that $e = (v_{n-1}, v_n)$ is an edge in X, then

$$W_X = W_{X \setminus e} - W_{X/e} \uparrow,$$

where $\uparrow$ is a linear operation defined on monomials with $(x_{i_1} x_{i_2} \dots x_{i_{n-2}} x_{i_{n-1}}) \uparrow = x_{i_1} x_{i_2} \dots x_{i_{n-2}} x_{i_{n-1}}^2$.



Digraphs X, $X \setminus (2,3)$ and $X/(2,3)$ respectively

Expansion in the power sum basis

Let $X = (V, E)$ be a digraph and let $\mathbb{S}_V$ be the group of permutations of V. We define

$$\mathbb{S}_V(X, \overline{X}) = \{\sigma \in \mathbb{S}_V \mid \text{ each cycle of } \sigma \text{ is a cycle of } X \text{ or a cycle of } \overline{X}\}.$$

If $X = (V, E)$ is a digraph with labeled vertices $v_1, v_2, \dots, v_n$, then

$$W_X = \sum_{\sigma \in \mathbb{S}_V(X, \overline{X})} (-1)^{\varphi(\sigma)} p_{\mathrm{Type}(\sigma)},$$

with $\varphi(\sigma) = \sum_\gamma (\ell(\gamma) - 1)$, where the summation runs over all cycles γ of σ that are cycles in X, $\ell(\gamma)$ denotes the length of the cycle γ and $\mathrm{Type}(\sigma)$ is the partition of V whose blocks correspond to the cycles of the unique cycle decomposition of σ.

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