Enumerating 1324-avoiders with few inversions

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Counting the 1324-avoiding permutations is notoriously difficult. We work towards this goal by enumerating $\mathrm{av}_n^k(1324)$, the number of 1324-avoiding n-permutations with exactly k inversions, for all k and $n \geqslant \frac{k+7}{2}$. The result follows from a new structural characterization of the permutations in question. Along the way, we prove half of a conjecture of Claesson, Jelínek and Steingrímsson (2012). Preprint: `arXiv:2408.15075`

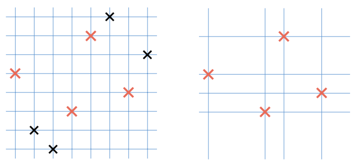
$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
1	1																									
2	1	1																								
3	1	2	2	1																						
4	1	2	5	6	5	3	1																			
5	1	2	5	10	16	20	20	15	9	4	1															
6	1	2	5	10	20	32	51	67	79	80	68	49	29	14	5	1										
7	1	2	5	10	20	36	61	96	148	208	268	321	351	347	308	241	165	98	49	20	6	1				
8	1	2	5	10	20	36	65	106	171	262	397	568	784	1019	1264	1478	1628	1681	1619	1441	1173	866	574	338	174	76
9	1	2	5	10	20	36	65	110	181	286	443	664	985	1416	1988	2715	3589	4579	5631	6654	7559	8225	8545	8457	7930	7006
10	1	2	5	10	20	36	65	110	185	296	467	714	1077	1582	2305	3284	4617	6374	8665	11521	15012	19067	23599	28426	33300	37862
11	1	2	5	10	20	36	65	110	185	300	477	738	1127	1682	2477	3584	5134	7240	10100	13915	18976	25563	34017	44640	57739	73421
12	1	2	5	10	20	36	65	110	185	300	481	748	1151	1732	2577	3768	5450	7766	10976	15312	21171	28973	39338	52919	70657	93482
13	1	2	5	10	20	36	65	110	185	300	481	752	1161	1756	2627	3868	5634	8098	11526	16216	22632	31266	42845	58213	78531	105137
14	1	2	5	10	20	36	65	110	185	300	481	752	1165	1766	2651	3918	5734	8282	11858	16786	23568	32768	45234	61902	84130	113477
15	1	2	5	10	20	36	65	110	185	300	481	752	1165	1770	2661	3942	5784	8382	12042	17118	24138	33728	46776	64346	87939	119306
16	1	2	5	10	20	36	65	110	185	300	481	752	1165	1770	2665	3952	5808	8432	12142	17302	24470	34298	47736	65916	90431	123184
17	1	2	5	10	20	36	65	110	185	300	481	752	1165	1770	2665	3956	5818	8456	12192	17402	24654	34630	48306	66876	92001	125708
18	1	2	5	10	20	36	65	110	185	300	481	752	1165	1770	2665	3956	5822	8466	12216	17452	24754	34814	48638	67446	92961	127278
19	1	2	5	10	20	36	65	110	185	300	481	752	1165	1770	2665	3956	5822	8470	12226	17476	24804	34914	48822	67778	93531	128238
20	1	2	5	10	20	36	65	110	185	300	481	752	1165	1770	2665	3956	5822	8470	12230	17486	24828	34964	48922	67962	93863	128808

Table 1. The numbers $\mathrm{av}_n^k(1324)$.

Preliminaries

A permutation $\pi \in S_n$ *contains* a pattern $p \in S_m$ if π has a subsequence that is order-isomorphic to p. Otherwise π *avoids* p. Let

$$\mathrm{Av}_n(p) = \{\pi \in S_n : \pi \text{ avoids } p\} \quad \text{and} \quad \mathrm{av}_n(p) = |\mathrm{Av}_n(p)|.$$



π contains 3142

π avoids 1324

We also set $\mathrm{Av}_n^k(p) = \{\pi \in \mathrm{Av}_n(p) : \mathrm{inv}(\pi) = k\}$ and $\mathrm{av}_n^k(p) = |\mathrm{Av}_n^k(p)|$.

Background

The study of pattern avoiding permutations was initiated by Knuth (1968) with his characterization of the *stack-sortable* permutations as the 231-avoiders. In fact, for any length 3 pattern p,

$$\mathrm{av}_n(p) = C_n = \frac{1}{n+1}\binom{2n}{n},$$

the n-th Catalan number. However, for length 4 patterns the situation is much more complicated. There are three essentially distinct cases:

Class	Enumeration	Asymptotics
$\mathrm{Av}(1234)$	Complicated formula by Gessel (1990)	9^n
$\mathrm{Av}(1342)$	Complicated formula by Bóna (1997)	8^n
$\mathrm{Av}(1324)$	?	$10.27^n < a_n < 13.5^n$

Moral. We know almost nothing about the 1324-avoiders.

One famous conjecture would provide an improvement.

Conjecture 1 (Claesson–Jelínek–Steingrímsson 2012). For all nonnegative integers n and k,

$$\mathrm{av}_n^k(1324) \leqslant \mathrm{av}_{n+1}^k(1324).$$

Remark. If Conjecture 1 is true, then

$$\mathrm{av}_n(1324) < C^n,$$

where $C = e^{\pi\sqrt{2/3}} < 13.002$.

Observation. Conjecture 1 $\iff$ the columns of Table 1 are increasing.

Main result

Many have tried to attack Conjecture 1, to no avail. It holds trivially in the light blue triangle of Table 1: the columns turn constant!

Our main result enumerates the values in the dark blue wedge of Table 1, thus proving Conjecture 1 for that region.

Theorem 2 (Linusson–V. 2025). For all nonnegative integers k and $n \geqslant \frac{k+7}{2}$,

$$\mathrm{av}_n^k(1324) = a(k) - 4a(k-n+1) - 6\sum_{i=0}^{k-n} a(i),$$

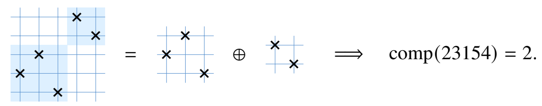
where $a(k) = \sum_{i=0}^{k} p(i)p(k-i)$ and $p(k)$ is the number of integer partitions of k. In particular,

$$\mathrm{av}_{n+1}^k(1324) - \mathrm{av}_n^k(1324) = 4a(k-n+1) + 2a(k-n) \geqslant 0,$$

and this difference has a combinatorial interpretation.

The ideas behind the proof

Permutations can be decomposed with respect to the *direct sum*:



If $\pi \in \mathrm{Av}(1324)$ is decomposable, it must be of the form

$$\pi = \sigma \oplus 1 \oplus 1 \oplus \ldots \oplus 1 \oplus \tau, \qquad \text{where } \sigma \in \mathrm{Av}(132),\ \tau \in \mathrm{Av}(213). \qquad (1)$$

An n-permutation with at most $n-2$ inversions is decomposable, so all permutations in the light triangle of Table 1 satisfy (1). Inversion sequences of 132-avoiders are partitions, so the sequence $1, 2, 5, 10, 20, 36, \ldots$ is $a(k) = \sum_{i=0}^{k} p(i)p(k-i)$.

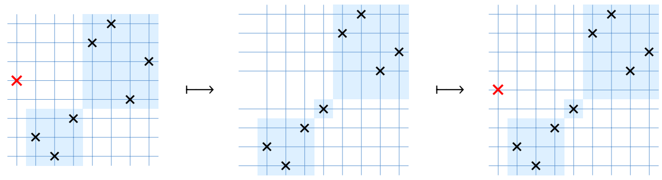
Question. 1324-avoiders with very few inversions have a very nice structure. What if we allow slightly more inversions?

Definition 3 (Linusson–V. 2025). An n-permutation π is called *almost decomposable* if it is indecomposable, but

$$\max\{\mathrm{comp}(\pi \setminus e) : e \in \{1, \pi_1, n, \pi_n\}\} \geqslant 2.$$

Theorem 4 (Linusson–V. 2025). If $n \geqslant \frac{k+7}{2}$, then all permutations in $\mathrm{Av}_n^k(1324)$ are decomposable or almost decomposable.

Theorem 4 allows us to build an injection $\mathrm{Av}_n^k(1324) \to \mathrm{Av}_{n+1}^k(1324)$ when $n \geqslant \frac{k+7}{2}$.



We can extend this mapping naturally to *all* almost decomposable 1324-avoiders using symmetries of the square. The rest of the proof of Theorem 2 consists of a careful enumeration of the collection

$$\mathcal{R}_n^k := \mathrm{Av}_{n+1}^k(1324) \setminus f\big(\mathrm{Av}_n^k(1324)\big),$$

where f is our injection. For example, any $\pi \in \mathrm{Av}_{n+1}^k(1324)$ with $\pi_1 = n+1$ is in $\mathcal{R}_n^k$.

Question. Can our method be extended to prove more of Conjecture 1, or to other problems in pattern avoidance?