Chow Polynomials of uniform Matroids

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Background

$\mathcal{M}$ matroid of rank $\mathrm{rk}(\mathcal{M})$ on ground set of size n

$\mathcal{L}_\mathcal{M}$ lattice of flats of $\mathcal{M}$ ← geometric lattice

The CHOW RING of $\mathcal{M}$:

$$A = A(\mathcal{M}) = A(\mathcal{L}_\mathcal{M}) = A(\mathcal{L}_\mathcal{M}, \mathcal{L}_\mathcal{M}\setminus\{\hat{0}\}) = \mathbb{Z}[x_F \mid F \in \mathcal{L}_\mathcal{M}\setminus\{\hat{0}\}] / (I+J)$$

max. building set

[FY04]: $A = A(\mathcal{L}, \mathcal{G})$ — finite, graded lattice; building set

$I, J \subset S$ both
- graded ideals
- homogeneous

[FY04] $= \bigoplus_{m=0}^{\mathrm{rk}(\mathcal{M})-1} A^m$ graded ring

The CHOW POLYNOMIAL of $\mathcal{M}$ is the Hilbert-Poincaré series of A:

$$H_\mathcal{M}(x) = \sum_{m=0}^{r-1} \dim_\mathbb{Q}(A^m)\cdot x^m$$

The AUGMENTED CHOW RING of $\mathcal{M}$

$$\tilde{A} = A(\mathcal{M}\times\{e\}, \tilde{\mathcal{G}}) \text{ is a Chow ring}$$

free coextension of $\mathcal{M}$; non-max. building set

$$= \bigoplus_{m=0}^{\mathrm{rk}(\mathcal{M})} \tilde{A}^m \quad \text{graded ring}$$

[ANR23, Remark 2.27] for overview on literature on $\tilde{A}$

The AUGMENTED CHOW POLYNOMIAL of $\mathcal{M}$ is the Hilbert-Poincaré series

$$\mathrm{aug}H_\mathcal{M}(x) = \sum_{m=0}^{\mathrm{rk}(\mathcal{M})} \dim_\mathbb{Q}(\tilde{A}^m)\cdot x^m$$

Combinatorial Properties of Chow polys

- ✓ unimodal } [AHK18]
- ✓ palindromic } [AHK18]
- ✓ γ-positive [FSMV24], [Stu24] and more
- ? log-concave
- ? real-rooted

real-rooted + non-neg. ⟹ log-concave ⟹(+positive) unimodal; real-rooted + non-neg. ⟹(+palindromic) γ-positive ⟹ unimodal, palindromic

$f(x) = \sum_{m=0}^{r} a_m x^m \in \mathbb{R}[x]$ is called
- unimodal if $a_0 \le a_1 \le \dots \le a_{\lfloor r/2 \rfloor} = a_{\lceil r/2 \rceil} \ge \dots \ge a_r$
- palindromic if $a_m = a_{r-m}$ for $m \le r/2$
- γ-positive if $f(x) = \sum_{i=0}^{\lfloor r/2 \rfloor} \gamma_i \cdot x^i \cdot (1+x)^{r-2i}$, $\gamma_i > 0$
- log-concave if $a_m^2 \ge a_{m-1}\cdot a_{m+1}$ for $1 \le m < r$
- real-rooted if f has only real roots

? Are Chow polynomials real-rooted? ← yes, for $H_{U_{k,n}}(x)$

Determine Chow polys

$\lambda: \mathcal{E}(\mathcal{L}) \to \mathbb{Z}_{>0}$ R-labeling of $\mathcal{L}$ (edges in Hasse-diagram)

$\lambda_\mathcal{C} = (\lambda_1, \lambda_2, \dots, \lambda_{\mathrm{rk}(\mathcal{L})})$, $\lambda_i = \lambda(C_{i-1} \lessdot C_i)$

$\mathcal{C} = \{C_0 \lessdot C_1 \lessdot \dots \lessdot C_{\mathrm{rk}(\mathcal{L})}\}$ maximal chain

$\mathrm{des}(\lambda_\mathcal{C}) = \#\{ i \in \{1, \dots, \mathrm{rk}(\mathcal{L})-1\} \mid \lambda_i > \lambda_{i+1} \}$

λ R-labeling: edge-labeling, s.t. every interval has unique maximal chain w/ increasing labels

total order $a_1 < a_2 < \dots < a_s$ on the atoms of $\mathcal{L}$ defines R-labeling λ by $\lambda(x \lessdot y) = \min\{i \mid x \vee a_i = y\}$

Theorem [Stu24]

$$H_\mathcal{M}(x) = \sum_{\substack{\mathcal{C} \text{ max. chain in } \mathcal{L}_\mathcal{M} \\ \lambda_i > \lambda_{i+1} \Rightarrow \lambda_{i+1} < \lambda_{i+2} \\ \lambda_1 < \lambda_2}} x^{\mathrm{des}(\lambda_\mathcal{C})}\cdot(1+x)^{\mathrm{rk}(\mathcal{M})-1-2\cdot\mathrm{des}(\lambda_\mathcal{C})}$$

one proof of the γ-positivity

Theorem [H. 24+]

$U_{k,n}$ uniform matroid on ground set $[n]$ of rank $k \le n$.

$$H_{U_{k,n}}(x) = \sum_{m=0}^{k-1} \#\left\{ \begin{array}{l} \text{loopless Schubert matroids on } [n] \\ \text{of rank } m+1 \text{ and cogirth} > n-k \end{array} \right\} \cdot x^m$$

cogirth of a matroid $\mathcal{M}$ is the size of the smallest circuit in $\mathcal{M}^*$

$$= \sum_{\substack{I \subseteq \{1,\dots,k\} \\ 1 \in I}} \binom{n}{\Delta I}\cdot x^{|I|-1}$$

multinomial coefficient

$$= \sum_{\substack{D \subseteq \{1,\dots,k-1\} \\ i \in D \Rightarrow i+1 \notin D \\ 1 \notin D}} E(n, D)\cdot x^{|D|}\cdot(1+x)^{k-1-2\cdot|D|}$$

Eulerian number $E(n, D) = \#\{ w \in \mathfrak{S}_n \mid \mathrm{Des}(w) = D \}$

$$= \sum_{\substack{\sigma \in \mathfrak{S}_k \\ \sigma(i) > \sigma(i+1) \Rightarrow \sigma(i+1) < \sigma(i+2) \\ \sigma(1) < \sigma(2)}} \binom{n-\sigma(k)}{k-\sigma(k)}\cdot x^{\mathrm{des}(\sigma)}\cdot(1+x)^{k-1-2\cdot\mathrm{des}(\sigma)}$$

Schubert matroid $S_{I,\pi}$ on $[n]$ (ground set), $I = \{i_1 < \dots < i_{m+1}\} \subseteq [n]$, $\pi \in \mathfrak{S}_n$

bases: $\left\{ \pi^{-1}(J) \;\middle|\; \begin{array}{l} J = \{j_1 < \dots < j_{m+1}\} \subseteq [n] \\ i_1 \le j_1, i_2 \le j_2, \dots, i_{m+1} \le j_{m+1} \end{array} \right\}$

$[S_I] = \{ S_{J,\pi} \mid J \in \binom{[n]}{\#I}, \pi \in \mathfrak{S}_n, S_{J,\pi} \cong S_{I,\mathrm{id}} \}$

$[S_I]$	⟷	I
contains loopless matroids	⟷	$1 \in I$
with cogirth $> n-k$	⟷	$I \subseteq \{1,\dots,k\}$

$I = I_1 \dot\cup I_2 \dot\cup \dots \dot\cup I_s \subseteq [k]$
- → partition into maximal consecutive subsets
- → $\min I_1 < \min I_2 < \dots < \min I_s$ (differences)

$$\binom{n}{\Delta I} = \frac{n!}{(\min I_1 - 1)!\cdot(\min I_2 - \min I_1)!\cdots(\min I_s - \min I_{s-1})!\,(n+1-\min I_s)!}$$

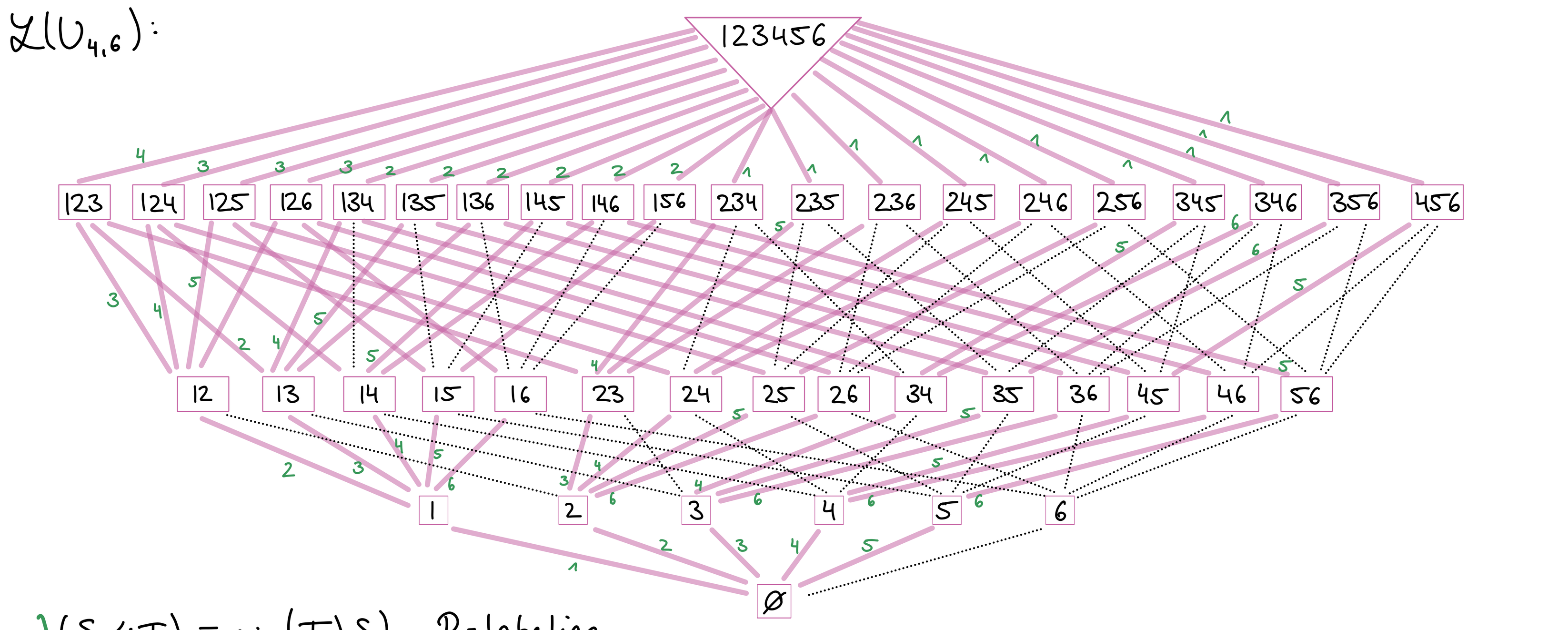
$I = \{2,3,5,7,8,9\} \subseteq [10]$ with $I_1 = \{2,3\}$, $I_2 = \{5\}$, $I_3 = \{7,8,9\}$

$$\binom{10}{\Delta I} = \frac{10!}{(2-1)!(5-2)!(7-5)!(11-7)!} = 12.600$$

! Analog theorems for $\mathrm{aug}H_\mathcal{M}(x)$ and $\mathrm{aug}H_{U_{k,n}}(x)$ exist! → less restrictions on range of sums; for γ-expansion and "Chow-in-4-ways"

Uniform matroid $U_{4,6}$

$\mathcal{L}(U_{4,6})$:



$\lambda(S \lessdot T) = \min(T \setminus S)$ R-labeling

$$H_{U_{4,6}}(x) = \sum_{\text{max. chains}} x^{\mathrm{des}(\lambda_\mathcal{C})}(1+x)^{2-2\cdot\mathrm{des}(\lambda_\mathcal{C})} = x^3 + 36x^2 + 36x + 1$$

$$\mathrm{aug}H_{U_{4,6}}(x) = x^4 + 42x^3 + 117x^2 + 42x + 1$$

$$H_{U_{4,6}}(x) = \underbrace{\binom{6}{\Delta\{1\}}}_{1} + \left[\underbrace{\binom{6}{\Delta\{1,2\}}}_{1} + \underbrace{\binom{6}{\Delta\{1,3\}}}_{15} + \underbrace{\binom{6}{\Delta\{1,4\}}}_{20}\right]\cdot x + \left[\underbrace{\binom{6}{\Delta\{1,2,3\}}}_{1} + \underbrace{\binom{6}{\Delta\{1,2,4\}}}_{20} + \underbrace{\binom{6}{\Delta\{1,3,4\}}}_{15}\right]\cdot x^2 + \underbrace{\binom{6}{\Delta[4]}}_{1} x^3$$

$$= \underbrace{E(6,\emptyset)}_{1}\cdot(1+x)^3 + \left[\underbrace{E(6,\{2\})}_{\binom{6}{2}-1} + \underbrace{E(6,\{3\})}_{\binom{6}{3}-1}\right]\cdot x\cdot(1+x)$$

$$= (1+x)^3 + \left[\binom{5}{3} + 2\cdot\binom{4}{2} + 3\cdot\binom{3}{1} + 2\cdot\binom{2}{0}\right]\cdot x\cdot(1+x)$$

$\sigma = \mathrm{id}$; $[2,3,4,1]$; $[1,2,4,3], [1,4,2,3], [2,4,1,3]$

$\mathcal{C} = \{\emptyset \lessdot 4 \lessdot 46 \lessdot 146 \lessdot 123456\}$

$\lambda_\mathcal{C} = (4, 6, 1, 2)$

extend: $[4,6,1,2,3,5] \in \mathfrak{S}_6$

standardize: $[3,4,1,2] \in \mathfrak{S}_4$

Uniform → any matroid

! Uniform matroids maximize Chow polynomials coefficient-wise [FMSV24] $H_{U_{k,n}}(x) - H_\mathcal{M}(x) \in \mathbb{N}[x]$

restrict the coefficients in the Theorem

? Can we write

$$H_\mathcal{M}(x) = \sum_{m \ge 0} \#\left\{ \begin{array}{l} \text{matroids on } [n] \text{ of rank } m+1 \\ \text{with "same restriction"} \end{array} \right\}\cdot x^m$$

$$H_\mathcal{M}(x) = \sum_D \#\{ w \in \mathfrak{S}_n \mid \mathrm{Des}(w) = D \text{ and "same restriction"} \}\cdot x^{|D|}\cdot(1+x)^{\mathrm{rk}(\mathcal{M})-1-2\cdot|D|}$$

References

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