Molecules of affine fixed-point-free W-graphs

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Affine permutations and involutions

- The *symmetric group* S_n is the group of bijections $\pi : [n] \to [n] := \{1, 2, \dots, n\}$.
- The *affine symmetric group* $\tilde{S}_n$ is the group of bijections $\pi : \mathbb{Z} \to \mathbb{Z}$ satisfying
$$\pi(i+n) = \pi(i) + n \quad \text{and} \quad \pi(1) + \pi(2) + \cdots + \pi(n) = 1 + 2 + \cdots + n.$$
The finite subgroup of elements $\pi \in \tilde{S}_n$ with $\pi([n]) = [n]$ may be identified with S_n.
- Let $s_i \in \tilde{S}_n$ be permutation interchanging $i \leftrightarrow i+1$, fixing all $j \notin \{i, i+1\} + n\mathbb{Z}$. Then $s_i = s_{i+n}$ and $\tilde{S}_n = \langle s_1, \dots, s_n \rangle$ is a Coxeter group with *length function* ℓ.
- An *affine involution* is $z \in \tilde{S}_n$ such that $z^2 = 1$. An *affine fixed-point-free involution* is an affine involution z such that there are no $x \in [n]$ with $z(x) = x$.
- The set of all involutions is denoted as $\tilde{I}_n$ while the set of all fixed-point-free involutions is denoted as $\mathcal{F}_n$. On $\mathcal{F}_n$, we define $\ell^{\mathsf{FPF}}(z) = \frac{1}{2}(\ell(z) - \frac{n}{2})$.
- Given $\pi \in \tilde{S}_n$, define $\beta(\pi) = \frac{1}{2n}\sum_{i=1}^{n} |\pi(i) - r_n(\pi(i))|$, where $r_n(i)$ for $i \in \mathbb{Z}$ denotes the unique element of $[n]$ that satisfies $r_n(i) \equiv i \pmod n$. For $z \in \mathcal{F}_n$, define $\mathrm{sgn}_{\mathsf{FPF}}(z) = (-1)^{\beta(z)}$.
- Let $\Theta^+ = s_1 s_3 \cdots s_{n-1} = [2, 1, \dots, n, n-1] \in \tilde{I}_n$ and $\Theta^- = s_2 s_4 \cdots s_n = [1, 0, \dots, n-1, n-2] \in \tilde{I}_n$, so that $\mathrm{sgn}_{\mathsf{FPF}}(\Theta^{\pm}) = \pm 1$. Define $\mathcal{F}_n^+$ as the $\tilde{S}_n$-conjugacy class of Θ^+ and $\mathcal{F}_n^-$ as the $\tilde{S}_n$-conjugacy class of Θ^-. One can show that
$$\mathcal{F}_n^+ = \{z \in \mathcal{F}_n : \mathrm{sgn}_{\mathsf{FPF}}(z) = 1\} \quad \text{and} \quad \mathcal{F}_n^- = \{z \in \mathcal{F}_n : \mathrm{sgn}_{\mathsf{FPF}}(z) = -1\} \quad (0.1)$$
and hence that $\mathcal{F}_n = \mathcal{F}_n^+ \sqcup \mathcal{F}_n^-$

Tabloids

- Let $\lambda = (\lambda_1, \dots, \lambda_\ell)$ be a partition of size $\sum_i \lambda_i \leq n$. A *tabloid* P of shape λ is an equivalence class of fillings of the Young diagram of shape λ with elements of $[\overline{n}]$ under identification of fillings that differ by reordering elements within rows. Here $\overline{i}$ denotes the equivalence class of integers $k \equiv i \pmod n$. We think of the i-th row of a tabloid P as a set $P_i \subseteq [\overline{n}]$.

Affine Matrix-Ball Construction

- Matrix-Ball Construction (MBC) is a construction algorithm of RSK correspondence besides the insertion algorithm.
- Chmutov, Pylyavskyy and Yuvidona generalized MBC to affine symmetric groups. They defined the Affine Matrix-Ball Construction (AMBC) mapping an affine permutation to the triplet of two tabloids (P, Q) with same shape and a dominant vector (ρ).
- For example, we have
$$[6,1,14,3,18,19,12,15,17,10] \mapsto \left(\begin{array}{|c|c|c|} \hline \overline{1} & \overline{3} & \overline{10} \\ \hline \overline{2} & \overline{5} & \overline{6} \\ \hline \overline{4} & \overline{7} & \overline{9} \\ \hline \overline{8} \\ \cline{1-1} \end{array}, \begin{array}{|c|c|c|} \hline \overline{3} & \overline{5} & \overline{6} \\ \hline \overline{7} & \overline{8} & \overline{9} \\ \hline \overline{1} & \overline{4} & \overline{10} \\ \hline \overline{2} \\ \cline{1-1} \end{array}, \begin{array}{|c|} \hline 2 \\ \hline 3 \\ \hline 1 \\ \hline 4 \\ \hline \end{array} \right).$$

Affine FPF graphs and molecules

- For an algebra $\mathcal{A}$, an I*-labeled graph* for a finite set I is a triple $\Gamma = (V, \omega, \nu)$ where (i) V is a finite vertex set; (ii) $\omega : V \times V \to \mathcal{A}$ is a map; (iii) $\nu : V \to \mathcal{P}(I)$ is a map assigning a subset of I to each vertex.
We view Γ as a weighted directed graph on the vertex set V with an edge $x \xrightarrow{\omega(x,y)} y$ if $\omega(x, y) \neq 0$.
- An S-labeled graph $\Gamma = (V, \omega, \nu)$ is a W*-graph* if the free $\mathcal{A}$-module generated by V can be given an $\mathcal{H}$-module structure with
$$H_s x = \begin{cases} vx & s \notin \nu(x) \\ -v^{-1}x + \sum_{y \in V; s \notin \nu(y)} \omega(x,y) y & s \in \nu(x) \end{cases} \quad \text{for } s \in S \text{ and } x \in V.$$
- Let $\mathcal{M} = \mathcal{A}\text{-span}\{M_z : z \in \mathcal{F}_n\}$ and $\mathcal{N} = \mathcal{A}\text{-span}\{N_z : z \in \mathcal{F}_n\}$ denote the free $\mathcal{A}$-modules with bases given by the symbols M_z and N_z for $z \in \mathcal{F}_n$. We call $\{M_z\}_{z \in \mathcal{F}_n}$ and $\{N_z\}_{z \in \mathcal{F}_n}$ the standard bases of $\mathcal{M}$ and $\mathcal{N}$, respectively.
- Proposition(Z.) Both $\mathcal{M}$ and $\mathcal{N}$ have unique $\mathcal{H}$-module structures such that if $s \in S$ and $z \in \mathcal{F}_n$ then we have
$$H_s M_z = \begin{cases} M_{szs} & \ell^{\mathsf{FPF}}(szs) > \ell^{\mathsf{FPF}}(z) \\ vM_z & \ell^{\mathsf{FPF}}(szs) = \ell^{\mathsf{FPF}}(z) \\ M_{szs} + (v - v^{-1})M_z & \ell^{\mathsf{FPF}}(szs) < \ell^{\mathsf{FPF}}(z) \end{cases}$$
and
$$H_s N_z = \begin{cases} N_{szs} & \ell^{\mathsf{FPF}}(szs) > \ell^{\mathsf{FPF}}(z) \\ -v^{-1}N_z & \ell^{\mathsf{FPF}}(szs) = \ell^{\mathsf{FPF}}(z) \\ N_{szs} + (v - v^{-1})N_z & \ell^{\mathsf{FPF}}(szs) < \ell^{\mathsf{FPF}}(z). \end{cases}$$
- Define $\mathbf{m}_{x,y}$ and $\mathbf{n}_{x,y}$ for $x, y \in \mathcal{F}_n$ as the polynomials in $\mathbb{Z}[v^{-1}]$ such that
$$\underline{M}_y = \sum_{x \in \tilde{S}_n} \mathbf{m}_{x,y} M_x \quad \text{and} \quad \underline{N}_y = \sum_{x \in \tilde{S}_n} \mathbf{n}_{x,y} N_x.$$
Let $\mu_{\mathbf{m}}(x, y)$ and $\mu_{\mathbf{n}}(x, y)$ denote the coefficients of v^{-1} in $\mathbf{m}_{x,y}$ and $\mathbf{n}_{x,y}$. Define $\nu_{\mathbf{m}}$, $\nu_{\mathbf{n}}$: $\mathcal{F}_n \to \mathcal{P}(S)$ by
$$\nu_{\mathbf{m}}(x) = \{s \in S : sxs \leq_F x\} \quad \text{and} \quad \nu_{\mathbf{n}}(x) = \{s \in S : x \leq_F sxs\}$$
where $S = \{s_1, s_2, \dots, s_n\} \subset \tilde{S}_n$. Finally, let $\omega_{\mathbf{m}}$: $\mathcal{F}_n \times \mathcal{F}_n \to \mathbb{Z}$ be the map with
$$\omega_{\mathbf{m}}(x, y) = \begin{cases} \mu_{\mathbf{m}}(x, y) + \mu_{\mathbf{m}}(y, x) & \nu_{\mathbf{m}}(x) \not\subset \nu_{\mathbf{m}}(y) \\ 0 & \nu_{\mathbf{m}}(x) \subset \nu_{\mathbf{m}}(y). \end{cases}$$
Define $\omega_{\mathbf{n}}$: $\mathcal{F}_n \times \mathcal{F}_n \to \mathbb{Z}$ by the same formula, but with $\mu_{\mathbf{m}}$ and $\nu_{\mathbf{m}}$ replaced by $\mu_{\mathbf{n}}$ and $\nu_{\mathbf{n}}$.

Proposition(Marberg)

Both $\Gamma_n^{\mathbf{m}} = (\mathcal{F}_n, \omega_{\mathbf{m}}, \nu_{\mathbf{m}})$ and $\Gamma_n^{\mathbf{n}} = (\mathcal{F}_n, \omega_{\mathbf{n}}, \nu_{\mathbf{n}})$ are $\tilde{S}_n$-graphs.

- We call these graphs affine **FPF** graphs. The strongly connected components in a W-graph Γ are called *cells*. The connected components with respect to bidirected edges are called *molecules*.

Theorem(Z.)

For two affine involutions w and v, they are in the same molecule in $\Gamma_n^{\mathbf{m}}$ only if they have the same sign, and corresponding to tabloids of the same shape with same dominant weight applying AMBC.

- This is just a necessary condition, which is not sufficient.
- For $n = 4$, by definition of molecule, we can find such two molecules:
$$\{[4,3,2,1], [-4,3,2,9], [3,-4,1,10], [-5,4,9,2], [4,-5,10,1], [4,11,-6,1]\}$$
and
$$\{[0,-1,6,5], [0,7,-2,5], [7,0,-3,6], [-1,8,5,-2], [8,-1,6,-3], [-8,-1,6,13]\}.$$
All of them have the same sign $+1$ and are corresponding to tabloids of the same shape $\begin{array}{|c|} \hline \\ \hline \\ \hline \\ \hline \\ \hline \end{array}$ with same dominant weight $\begin{array}{|c|} \hline 0 \\ \hline 0 \\ \hline 0 \\ \hline 0 \\ \hline \end{array}$.

Theorem(Z.)

The molecules of $\Gamma_n^{\mathbf{m}}$ with same sign, same shape and same dominant weight are isomorphic to each other.

- Denote the number of molecules of $\Gamma_n^{\mathbf{m}}$ with the same shape λ and same dominant weight ρ by $o(\lambda)$ and call it the order of λ. Recall that $a|b$ means a divides b for two integers a, b. Moreover, $a|b|c$ means $a|b$ and $b|c$.

Proposition(Z.) $o(\lambda)$ is independent of ρ. Moreover, we have $2|o(\lambda)|n$. Specifically, we have $o((1, 1, \cdots, 1)) = n$ and $o((\frac{n}{2}, \frac{n}{2})) = 2$.

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