A conjectural basis for the $(1,2)$-bosonic-fermionic coinvariant ring

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Abstract

We give a conjectural construction of a monomial basis for the coinvariant ring $R_n^{(1,2)}$, for the symmetric group $\mathfrak{S}_n$ acting on one set of bosonic (commuting) and two sets of fermionic (anticommuting) variables. Our construction interpolates between the modified Motzkin path basis for $R_n^{(0,2)}$ of Kim and Rhoades [2022] and the super-Artin basis for $R_n^{(1,1)}$ conjectured by Sagan and Swanson [2024] and proven by Angarone et al. [2024]. We prove that our proposed basis has cardinality $2^{n-1}n!$, and show how it gives a combinatorial expression for the Hilbert series. We also conjecture a Frobenius series for $R_n^{(1,2)}$.

Background

The classical coinvariant ring $R_n^{(1,0)} = \mathbb{C}[\boldsymbol{x}_n]/\langle \mathbb{C}[\boldsymbol{x}_n]_+^{\mathfrak{S}_n}\rangle$ is the quotient of a polynomial ring in n variables $\boldsymbol{x}_n = \{x_1, \ldots, x_n\}$ by $\mathfrak{S}_n$-invariant polynomials with no constant term. It has dimension $n!$, Hilbert series $[n]_q!$, and Frobenius series $\sum_{\lambda \vdash n} \sum_{T \in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}(T)} s_\lambda$. An important basis of the classical coinvariant ring is the Artin basis.

Haiman [1994] introduced the diagonal coinvariant ring $R_n^{(2,0)} = \mathbb{C}[\boldsymbol{x}_n, \boldsymbol{y}_n]/\langle \mathbb{C}[\boldsymbol{x}_n, \boldsymbol{y}_n]_+^{\mathfrak{S}_n}\rangle$, where $\mathfrak{S}_n$ acts diagonally by permuting the indices of the variables. Haiman [2002] proved that $R_n^{(2,0)}$ has dimension $(n+1)^{n-1}$, Hilbert series $\langle \nabla_{q,t}(e_n), h_1^n\rangle$, and Frobenius series $\nabla_{q,t}(e_n)$. Haglund and Loehr [2005] conjectured a combinatorial formula for its Hilbert series, which was proven when Carlsson and Mellit [2018] gave a combinatorial formula for $\nabla_{q,t}(e_n)$. A monomial basis was given by Carlsson and Oblomkov [2018].

Recently, there has been interest (see Bergeron [2020]) in extending the setting to include coinvariant rings with k sets of n commuting variables $\boldsymbol{x}_n, \boldsymbol{y}_n, \boldsymbol{z}_n, \ldots$ and j sets of n anticommuting variables $\boldsymbol{\theta}_n, \boldsymbol{\xi}_n, \boldsymbol{\rho}_n, \ldots$ (which commute with $\boldsymbol{x}_n, \boldsymbol{y}_n, \boldsymbol{z}_n, \ldots$). We denote this by

$$R_n^{(k,j)} = \mathbb{C}[\underbrace{\boldsymbol{x}_n, \boldsymbol{y}_n, \boldsymbol{z}_n, \ldots}_{k}, \underbrace{\boldsymbol{\theta}_n, \boldsymbol{\xi}_n, \boldsymbol{\rho}_n, \ldots}_{j}] \Big/ \langle \mathbb{C}[\underbrace{\boldsymbol{x}_n, \boldsymbol{y}_n, \boldsymbol{z}_n, \ldots}_{k}, \underbrace{\boldsymbol{\theta}_n, \boldsymbol{\xi}_n, \boldsymbol{\rho}_n, \ldots}_{j}]_+^{\mathfrak{S}_n}\rangle.$$

Then $R_n^{(k,j)}$ is a $GL_k \times GL_j \times \mathfrak{S}_n$-module.

Modified Motzkin Path Basis for $R_n^{(0,2)}$

The set of *modified Motzkin paths* of length n, $\Pi(n)_{>0}$, is the set of all paths $\pi = (p_1, \ldots, p_n)$ in $\mathbb{Z}^2$ where p_i is:

(a) an up-step $(1,1)$,
(b) a horizontal step $(1,0)$ with decoration θ_i,
(c) a horizontal step $(1,0)$ with decoration ξ_i,
(d) or a down-step $(1,-1)$ with decoration $\theta_i\xi_i$,

where the first step must be an up-step, and subsequently, the path never goes below the horizontal line $y=1$.

Define the *weight* $\mathrm{wt}(p_i)$ of a step p_i of a modified Motzkin path to be its decoration, or 1 if it does not have one. Define the *weight* $\mathrm{wt}(\pi)$ of a modified Motzkin path the product of the weights of each step.

Definition (Kim and Rhoades [2022]). The *modified Motzkin path basis* $B_n^{(0,2)}$ is the set of all weights of the modified Motzkin paths $\pi \in \Pi(n)_{>0}$, that is,

$$B_n^{(0,2)} := \{\mathrm{wt}(\pi) \mid \pi \in \Pi(n)_{>0}\}.$$

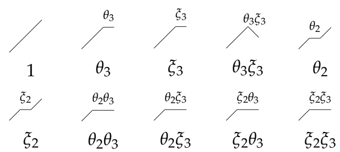

Figure 1: The basis $B_3^{(0,2)}$.

Theorem (Kim and Rhoades [2022]). *The modified Motzkin path basis basis $B_n^{(0,2)}$ is a basis for $R_n^{(0,2)}$.*

Super-Artin Basis for $R_n^{(1,1)}$

Let $\chi(P)$ be 1 if P is true, and 0 otherwise. Let θ_T denote the ordered product $\theta_{t_1} \cdots \theta_{t_k}$ for $T = \{t_1 < \cdots < t_k\}$. For any $T \subseteq \{2, \ldots, n\}$, define the *$\alpha$-sequence* $\alpha(T) = (\alpha_1(T), \ldots, \alpha_n(T))$ recursively by the initial condition $\alpha_1(T) = 0$ and for $2 \le i \le n$,

$$\alpha_i(T) = \alpha_{i-1}(T) + \chi(i \notin T).$$

Definition (Sagan and Swanson [2024]). The super-Artin set is

$$B_n^{(1,1)} := \{x^\alpha \theta_T \mid T \subseteq \{2, \ldots, n\} \text{ and } \alpha \le \alpha(T) \text{ componentwise}\}.$$

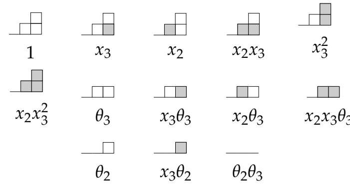

Figure 2: The basis $B_3^{(1,1)}$.

Theorem (Angarone et al. [2024]). *The super-Artin set $B_n^{(1,1)}$ is a basis for $R_n^{(1,1)}$.*

Conjectural Basis for $R_n^{(1,2)}$

Let ξ_S denote the ordered product $\xi_{s_1} \cdots \xi_{s_k}$ for $S = \{s_1 < \cdots < s_k\}$. For any $T, S \subseteq \{2, \ldots, n\}$, define the *generalized α-sequence* $\alpha(T,S) = (\alpha_1(T,S), \ldots, \alpha_n(T,S))$ recursively by the initial condition $\alpha_1(T,S) = 0$ and for $2 \le i \le n$,

$$\alpha_i(T,S) = \alpha_{i-1}(T,S) - 1 + \chi(i \notin T) + \chi(i \notin S).$$

Definition. We let

$$B_n^{(1,2)} := \{x^\alpha \theta_T \xi_S \mid \theta_T \xi_S \in B_n^{(0,2)} \text{ and } 0 \le \alpha_i \le \alpha_i(T,S) \,\forall\, 1 \le i \le n\}.$$

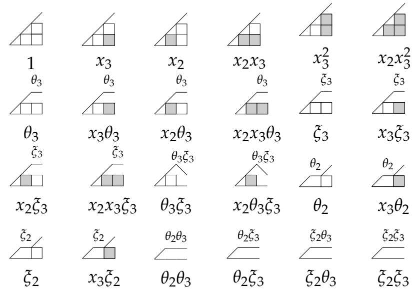

Figure 3: The basis $B_3^{(1,2)}$.

Conjecture. *The set $B_n^{(1,2)}$ is a basis for $R_n^{(1,2)}$.*

Define

$$\mathrm{stair}_q(\pi) := \prod_{k \in \alpha(T(\pi), S(\pi))} [k+1]_q,$$

where $T(\pi)$ and $S(\pi)$ consist of the indices of θ_i and ξ_i in the weight of π. Let $\deg_\theta(\pi) = |T(\pi)|$ and $\deg_\xi(\pi) = |S(\pi)|$.

Proposition. *Assuming the Conjecture,*

$$\mathrm{Hilb}(R_n^{(1,2)}; q; u, v) = \sum_{\pi \in \Pi(n)_{>0}} u^{\deg_\theta(\pi)} v^{\deg_\xi(\pi)} \mathrm{stair}_q(\pi).$$

We prove that $B_n^{(1,2)}$ has Zabrocki's conjectured dimension.

Theorem. *The cardinality of $B_n^{(1,2)}$ is $2^{n-1}n!$.*

The *fundamental quasisymmetric function* $Q_{S,n}$ is defined by

$$Q_{S,n} = \sum_{\substack{a_1 \le a_2 \le \cdots \le a_n, \\ a_i < a_{i+1} \text{ if } i \in S}} z_{a_1} z_{a_2} \cdots z_{a_n},$$

for a subset $S \subseteq \{1, \ldots, n-1\}$. For any $b \in B_n^{(1,2)}$, write

$$b = \pm \prod_{i=1}^n x_i^{\alpha_i} \theta_i^{\beta_i} \xi_i^{\gamma_i}.$$

Define i to be an *ascent of* b if one of the following occurs:

- $\beta_i < \beta_{i+1}$;
- $\beta_i = \beta_{i+1} = 1$ and $\alpha_i \ge \alpha_{i+1} + \gamma_{i+1}$; or
- $\beta_i = \beta_{i+1} = 0$ and $\alpha_i < \alpha_{i+1} + \gamma_{i+1}$.

For $b \in B_n^{(1,2)}$, we say that

$$\mathrm{Asc}(b) := \{i \in \{1, \ldots, n-1\} \mid i \text{ is an ascent of } b\}.$$

Conjecture.

$$\mathrm{Frob}(R_n^{(1,2)}; q; u, v) = \sum_{b \in B_n^{(1,2)}} u^{\deg_\theta(b)} v^{\deg_\xi(b)} q^{\deg_x(b)} Q_{\mathrm{Asc}(b),n}.$$

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Acknowledgments: Thanks to François Bergeron, Sylvie Corteel, Nicolle González, Sean Griffin, Mark Haiman, Brendon Rhoades, Josh Swanson, and Mike Zabrocki for invaluable conversations. Partial support was provided by the National Science Foundation Graduate Research Fellowship DGE-2146752.