Higher Rank Macdonald Polynomials

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Introduction

The study of **Macdonald polynomials** has been a central subject in algebraic combinatorics since their introduction by Macdonald [6]. Cherednik [3] showed that the study of Macdonald polynomials is intimately tied to the structure of **double affine Hecke algebras**. In particular, **non-symmetric** Macdonald polynomials $E_\mu(x_1,\cdots,x_n;q,t)$ may defined as (suitably normalized) **weight vectors** for certain operators Y_i on the polynomial ring $\mathbb{Q}(q,t)[x_1^{\pm 1},\cdots,x_n^{\pm 1}]$. Here we study a **higher rank** generalization of Macdonald polynomials $E_{\mu^\bullet}(x_{1,1},\cdots,x_{1,n},\cdots,x_{r,1},\cdots,x_{r,n};q_1,\cdots,q_r,t)$ in multiple sets of variables and their **symmetric**, by which we mean Hecke-invariant, analogues. We prove that many of the desirable properties of usual Macdonald polynomials hold in greater generality. This work generalizes the **vector-valued Macdonald polynomials** of Dunkl–Luque [4] as well as the author's prior work [1].

Set-Up

Let $t,q_1,q_2,q_3,\cdots$ be a family of algebraically independent commuting free variables. We work over the field $\mathbb{F}:=\mathbb{Q}(t,q_1,q_2,q_3,\cdots)$. Let $e_1,\cdots,e_n$ denote the standard coordinate basis vectors of $\mathbb{Z}^n$. For $n\geq 1$, $r\geq 1$ we define the space of ***rank r Laurent polynomials*** by $\mathbb{F}[x_{1,1}^{\pm 1},\cdots x_{1,n}^{\pm 1},\cdots,x_{r,1}^{\pm 1},\cdots x_{r,n}^{\pm 1}]=\mathbb{F}[\underline{x}_1,\cdots,\underline{x}_r]$ where we use the shorthand $\underline{x}_i=(x_{i,1}^{\pm 1},\cdots,x_{i,n}^{\pm 1})$. Further, for $\alpha\in\mathbb{Z}^n$ we write $\underline{x}_i^\alpha:=x_{i,1}^{\alpha_1}\cdots x_{i,n}^{\alpha_n}$. We consider the natural r-dimensional grading on $\mathbb{F}[\underline{x}_1,\cdots,\underline{x}_r]$ given by $\deg(\underline{x}_1^{\alpha^{(1)}}\cdots\underline{x}_r^{\alpha^{(r)}}):=(|\alpha^{(1)}|,\cdots,|\alpha^{(r)}|)$ where $|\alpha^{(i)}|:=\alpha_1^{(i)}+\cdots+\alpha_n^{(i)}\in\mathbb{Z}$. For every $1\leq i\leq r$ and $1\leq j\leq n-1$ we write $\xi_j:\mathbb{F}[\underline{x}_i]\to\mathbb{F}[\underline{x}_i]$ for the operator

$$\xi_j(\underline{x}_i^\alpha):=x_{i,j}\frac{\underline{x}_i^\alpha-\underline{x}_i^{s_j(\alpha)}}{x_{i,j}-x_{i,j+1}}.$$

Similarly, we let $(\mathfrak{S}_n)^r$ act on $\mathbb{F}[\underline{x}_1,\cdots,\underline{x}_r]$ by permuting the indices of the variables. In particular, we write s_j for the simple transposition $s_j=(j\ \ j+1)$.

Define operators $T_1,\cdots,T_{n-1},X_1^{\pm 1},\cdots,X_n^{\pm 1},\pi^{\pm 1}\in\mathrm{End}\left(\mathbb{F}[\underline{x}_1,\cdots,\underline{x}_r]\right)$ by

- $T_j:=s_j^{\otimes r}+(1-t)\sum_{k=0}^{r-1}s_j^{\otimes k}\otimes\xi_j\otimes 1^{\otimes(r-1-k)}$.
- $X_i(\underline{x}_1^{\mu^{(1)}}\cdots\underline{x}_r^{\mu^{(r)}}):=x_{1,i}\cdot\underline{x}_1^{\mu^{(1)}}\cdots\underline{x}_r^{\mu^{(r)}}$
- $\pi(\underline{x}_1^{\mu^{(1)}}\cdots\underline{x}_r^{\mu^{(r)}}):=q_1^{-\mu_n^{(1)}}\cdots q_r^{-\mu_n^{(r)}}x_{1,1}^{\mu_n^{(1)}}x_{1,2}^{\mu_1^{(1)}}\cdots x_{1,n}^{\mu_{n-1}^{(1)}}\cdots x_{r,1}^{\mu_n^{(r)}}x_{r,2}^{\mu_1^{(r)}}\cdots x_{r,n}^{\mu_{n-1}^{(r)}}$.

Example

$$T_1(\underline{\mathrm{x}}_1^{2e_2}\,\underline{\mathrm{x}}_2^{e_1-e_2}\,\underline{\mathrm{x}}_3^{-e_2+2e_4})=\underline{\mathrm{x}}_1^{2e_1}\,\underline{\mathrm{x}}_2^{e_2-e_1}\,\underline{\mathrm{x}}_3^{-e_1+2e_4}$$
$$+(1-t)(\underline{\mathrm{x}}_1^{2e_1}\,\underline{\mathrm{x}}_2^{e_2-e_1}\,\underline{\mathrm{x}}_3^{-e_2+2e_4}+\underline{\mathrm{x}}_1^{2e_1}\,\underline{\mathrm{x}}_2^{0}\,\underline{\mathrm{x}}_3^{-e_2+2e_4}+\underline{\mathrm{x}}_1^{2e_1}\,\underline{\mathrm{x}}_2^{e_1-e_2}\,\underline{\mathrm{x}}_3^{-e_2+2e_4}+\underline{\mathrm{x}}_1^{2e_1}\,\underline{\mathrm{x}}_2^{e_1-e_2}\,\underline{\mathrm{x}}_3^{-e_2+2e_4})$$

Define the **Cherednik operators** $Y_1^{\pm 1},\cdots,Y_n^{\pm 1}\in\mathrm{End}\left(\mathbb{F}[\underline{x}_1,\cdots,\underline{x}_r]\right)$ by

$$Y_i:=t^{n-i}T_{i-1}\cdots T_1\pi T_{n-1}^{-1}\cdots T_i^{-1}.$$

Theorem 1 [BW 25]

The space $\mathbb{F}[\underline{x}_1,\cdots,\underline{x}_r]$ along with the action by the operators $T_j,X_i^{\pm 1},Y_i^{\pm 1}$ defines a representation $V^{(r)}(q_1,\cdots,q_r)$ for the type GL_n double affine Hecke algebra $\mathfrak{D}_n(q_1)$ with parameters (q_1,t). Furthermore, $V^{(r)}(q_1,\cdots,q_r)$ is Y-semisimple with **one-dimensional** Y-weight spaces.

Higher Rank Non-Symmetric Macdonald Polynomials

Let $\widehat{\mathfrak{S}}_n:=\mathfrak{S}_n\ltimes\mathbb{Z}^n$ denote the **extended affine symmetric group**. We embed $\mathbb{Z}^n$ (non-trivially) as the set of minimal length coset representatives $\widehat{\mathfrak{S}}_n/\mathfrak{S}_n$. For $\ell\geq 1$ and $\sigma\in\widehat{\mathfrak{S}}_n$ we define $\Psi_{q_\ell}^\sigma:(\mathbb{F}^*)^n\to(\mathbb{F}^*)^n$ by the following:

- $\Psi_{q_\ell}^1(\alpha)=\alpha$
- $\Psi_{q_\ell}^{s_i}(\alpha)=(\cdots,\alpha_{i+1},\alpha_i,\cdots)$
- $\Psi_{q_\ell}^\pi(\alpha)=(q_\ell^{-1}\alpha_n,\alpha_1,\cdots,\alpha_{n-1})$
- $\Psi_{q_\ell}^{\sigma\gamma}=\Psi_{q_\ell}^\sigma\Psi_{q_\ell}^\gamma$.

For $\mu^\bullet=(\mu^{(1)},\cdots,\mu^{(r)})\in(\mathbb{Z}^n)^r$ define $\alpha_{\mu^\bullet}\in(\mathbb{F}^*)^n$ by $\alpha_{\mu^\bullet}:=\Psi_{q_1}^{\mu^{(1)}}\cdots\Psi_{q_r}^{\mu^{(r)}}(t^{n-1},\cdots,t,1)$. For $\mu\in\mathbb{Z}^n$ define T_μ inductively via $T_{(0,\cdots,0)}:=1$, $T_{s_j(\mu)}:=T_jT_\mu$ if $\mu_j>\mu_{j+1}$, and $T_{\pi(\mu)}:=\pi T_\mu$. For all $r\geq 1$, the set $\{\alpha_{\mu^\bullet}|\mu^\bullet\in(\mathbb{Z}^n)^r\}$ are the Y-weights of $V^{(r)}(q_1,\cdots,q_r)$.

Theorem 2 [BW 25]

There exists a *unique* family of higher rank Laurent polynomials

$$\{E_{\mu^{(1)},\cdots,\mu^{(r)}}(\underline{x}_1,\cdots,\underline{x}_r;q_1,\cdots,q_r,t)|\ r\geq 1\ ,\ (\mu^{(1)},\cdots,\mu^{(r)})\in(\mathbb{Z}^n)^r\}$$

satisfying the following properties:

- For $r=1$ and $\mu\in\mathbb{Z}^n$, $E_\mu(\underline{x}_1;q_1,t)=E_\mu(x_{1,1},\cdots,x_{1,n};q_1,t)$ is the usual non-symmetric Macdonald polynomial for μ following the conventions of Haiman–Haglund–Loehr [5].
- Each $E_{\mu^{(1)},\cdots,\mu^{(r)}}(\underline{x}_1,\cdots,\underline{x}_r;q_1,\cdots,q_r,t)\in\mathrm{V}^{(r)}(q_1,\cdots,q_r)$ is a Y-weight vector with weight $\alpha_{\mu^{(1)},\cdots,\mu^{(r)}}$.
- For every $\mu\in\mathbb{Z}^n$ and $(\beta^{(1)},\cdots,\beta^{(r)})\in(\mathbb{Z}^n)^r$, we have for some higher rank Laurent polynomials g_γ depending on μ and $(\beta^{(1)},\cdots,\beta^{(r)})$ the *triangular* expansion

$$\begin{aligned}&E_{\mu,\beta^{(1)},\cdots,\beta^{(r)}}(\underline{x}_1,\cdots,\underline{x}_{r+1};q_1,\cdots,q_{r+1},t)\\&=\underline{x}_1^\mu T_\mu E_{\beta^{(1)},\cdots,\beta^{(r)}}(\underline{x}_2,\cdots,\underline{x}_{r+1};q_2,\cdots,q_{r+1},t)+\sum_{\gamma<\mu}\underline{x}_1^\gamma g_\gamma(\underline{x}_2,\cdots,\underline{x}_{r+1}).\end{aligned}$$

Example

$$E_{(0,1,0),(2,1,0)}(\underline{\mathrm{x}}_1,\underline{\mathrm{x}}_2;q_1,q_2,t)=x_{1,2}x_{2,1}^2x_{2,2}+(t-1)x_{1,2}x_{2,1}^2x_{2,3}+\left(\frac{1-t}{1-q_2t^2}\right)x_{1,2}x_{2,1}x_{2,2}x_{2,3}$$
$$+\left(\frac{1-t}{1-q_1q_2^{-2}t^{-2}}\right)x_{1,1}x_{2,2}^2x_{2,3}+\left(\frac{1-t}{1-q_1q_2^{-2}t^{-2}}\right)\left(\frac{1-t}{1-q_2t^2}\right)x_{1,1}x_{2,1}x_{2,2}x_{2,3}$$

has Y-weight $\alpha_{(0,1,0),(2,1,0)}=(q_2^{-2},q_1^{-1}t^2,q_2^{-1}t)$.

Properties

For all $\mu^\bullet\in(\mathbb{Z}^n)^r$ Let $(\mu^{(1)},\cdots,\mu^{(r)})\in(\mathbb{Z}^n)^r$, $1\leq j\leq n-1$, and $1\leq\ell\leq r-1$ such that $s_j(\mu^{(i)})=\mu^{(i)}$ for $1\leq i\leq\ell-1$ and $s_j(\mu^{(\ell)})>\mu^{(\ell)}$, The following hold:

- $E_{\pi(\mu^{(1)}),\mu^{(2)},\cdots,\mu^{(r)}}=q_1^{\mu_n^{(1)}}X_1\pi E_{\mu^\bullet}$
- $E_{\mu^{(1)},\cdots,\mu^{(\ell-1)},s_j(\mu^{(\ell)}),\mu^{(\ell+1)},\cdots,\mu^{(r)}}=\left(T_j+\frac{t-1}{1-\frac{\alpha_{\mu^\bullet}(j)}{\alpha_{\mu^\bullet}(j+1)}}\right)E_{\mu^\bullet}$.

The above are higher rank analogues of the **Knop-Sahi relations**.

Furthermore, the stability properties hold:

- $E_{0,\mu^\bullet}(\underline{x}_1,\cdots,\underline{x}_{r+1};q_1,\cdots,q_{r+1},t)=E_{\mu^\bullet}(\underline{x}_2,\cdots,\underline{x}_{r+1};q_2,\cdots,q_{r+1},t)$
- $E_{\mu^\bullet,0}(\underline{x}_1,\cdots,\underline{x}_{r+1};q_1,\cdots,q_{r+1},t)=E_{\mu^\bullet}(\underline{x}_1,\cdots,\underline{x}_r;q_1,\cdots,q_r,t)$.

Higher Rank Symmetric Macdonald Polynomials

Denote by $\Phi_n^{(r)}$, the set of all $(\nu^{(1)},\cdots,\nu^{(r)})\in(\mathbb{Z}^n)^r$ such that $\nu^{(1)}$ is weakly decreasing and for all $1\leq j\leq r-1$ whenever $\nu_i^{(j)}=\nu_{i+1}^{(j)}$ we have $\nu_i^{(j+1)}\geq\nu_{i+1}^{(j+1)}$. We write $\Phi_{n,+}^{(r)}:=\Phi_n^{(r)}\cap(\mathbb{Z}_{\geq 0}^n)^r$. For $\nu^\bullet\in\Phi_n^{(r)}$ we define the ***rank r symmetric Macdonald polynomial*** $P_{\nu^\bullet}$ as an explicit scalar multiple of $\epsilon^{(n)}(E_{\nu^\bullet})$.

Define the space $W_n^{(r)}(q_1,\cdots,q_r):=\epsilon^{(n)}\left(V_{n,+}^{(r)}(q_1,\cdots,q_r)\right)$ of rank r symmetric polynomials with non-negative degree. Here symmetric means **Hecke-invariant**. Write $\Delta_n:=\epsilon^{(n)}\left((Y_1+\cdots+Y_n)-(1+t+\cdots+t^{n-1})\right)\epsilon^{(n)}$.

Theorem 3 [BW 25]

The set of higher rank symmetric Macdonald polynomials $\{P_{\nu^\bullet}\}_{\nu^\bullet\in\Phi_{n,+}^{(r)}}$ form a Δ_n-weight basis for $W_n^{(r)}(q_1,\cdots,q_r)$ with **distinct spectrum**.

Define the projection maps on higher rank polynomials

$$\Pi^{(n)}(\underline{\mathrm{x}}_1^{\alpha^{(1)}}\cdots\underline{\mathrm{x}}_r^{\alpha^{(r)}}):=\begin{cases}\underline{\mathrm{x}}_1^{(\alpha_1^{(1)},\cdots,\alpha_n^{(1)})}\cdots\underline{\mathrm{x}}_r^{(\alpha_1^{(r)},\cdots,\alpha_n^{(r)})} & \alpha_{n+1}^{(1)}=\cdots=\alpha_{n+1}^{(r)}=0\\ 0 & \text{otherwise.}\end{cases}$$

Define $W^{(r)}(q_1,\cdots,q_r):=\lim_{\leftarrow}W_n^{(r)}(q_1,\cdots,q_r)$ as the (r-dimensionally) **graded inverse limit** along the maps $\Pi^{(n)}$ and write $\Delta:=\lim_{\leftarrow}\Delta_n$.
We may naturally include sets as $\Phi_{n,+}^{(r)}\to\Phi_{n+1,+}^{(r)}$ and define $\Phi^{(r)}:=\lim_{\to}\Phi_{n,+}^{(r)}$ as the directed union. We represent elements $\nu^\bullet\in\Phi^{(r)}$ as

$$\nu^\bullet=(\nu_1^{(1)},\cdots,\nu_{n_1}^{(1)}|\cdots|\nu_1^{(r)},\cdots,\nu_{n_r}^{(r)})$$

for $n_j\geq 0$ with $\nu_{n_j}^{(j)}\neq 0$ and write $\ell(\nu^\bullet)=\max\{n_1,\cdots,n_r\}$. Given $\nu^\bullet\in\Phi^{(r)}$ and $n\geq\ell(\nu^\bullet)$ we write $\iota_n(\nu^\bullet):=(\nu_1^{(1)},\cdots,\nu_{n_1}^{(1)},0,\cdots,0|\cdots|\nu_1^{(1)},\cdots,\nu_{n_r}^{(1)},0,\cdots,0)\in\Phi_{n,+}^{(r)}$.
Using techniques similar to those of Schiffmann–Vasserot [7] and the author's prior work [1] we show the following:

Theorem 4 [BW 25]

For all $\nu^\bullet\in\Phi_{n+1,+}^{(r)}$,

$$\Pi^{(n)}(P_{\nu^\bullet})=\begin{cases}P_{(\nu_1^{(1)},\cdots,\nu_n^{(1)}),\cdots,(\nu_1^{(r)},\cdots,\nu_n^{(r)})} & \nu_{n+1}^{(1)}=\cdots=\nu_{n+1}^{(r)}=0\\ 0 & \text{otherwise.}\end{cases}$$

Furthermore, the space $W^{(r)}(q_1,\cdots,q_r)$ admits an action by the **positive elliptic Hall algebra** $\mathcal{E}^+$ of Burban-Schiffmann [2]. For all $r\geq 1$, $W^{(r)}(q_1,\cdots,q_r)$ is a graded $\mathcal{E}^+$-module with simple Δ-spectrum. The Δ-weight vectors are given for $\nu^\bullet\in\Phi^{(r)}$ by the rank r symmetric Macdonald functions $\mathcal{P}_{\nu^\bullet}:=\lim_n P_{\nu^\bullet}$.

References

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