Orthogonal roots, Macdonald representations, and quasiparabolic W-sets

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Background: a Weyl group action on "n-roots"

- Let Φ be an irreducible root system with simple roots Δ, positive roots Φ_+, and rank $n > 1$.

  For each $\alpha \in \Phi$, denote the corresponding reflection by s_α.

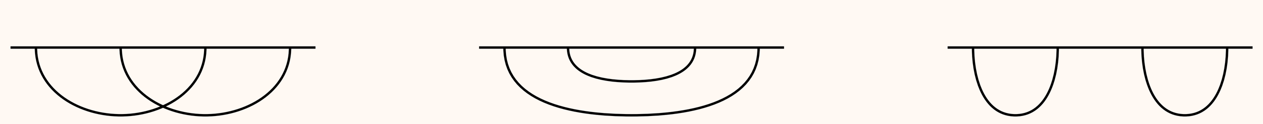
  Let $W = \langle s_\alpha : \alpha \in \Phi \rangle$ be the Weyl group of Φ, and let $S = \{s_\alpha : \alpha \in \Pi\}$ be the simple reflections of W.
- We define a *positive n-root of* Φ to be a set of n mutually orthogonal positive roots of Φ. Such a set exists if and only if Φ has Dynkin type E_7, E_8, or D_n for n even. We will focus on these types in this poster, and we denote the set of positive n-roots by Φ_+^n.
- The Weyl group W naturally acts on Φ_n^+ via the formula
  $$w \cdot R = \{|w \cdot \alpha| : \alpha \in R\},$$
  where $|\beta|$ is the unique positive root in $\{\beta, -\beta\}$.
- Meanwhile, each $R \in \Phi_+^n$ naturally gives rise to an element
  $$\gamma_R := \prod_{\alpha \in R} \alpha \in \mathcal{M} \subseteq \mathrm{Sym}(V),$$
  where $\mathcal{M}$ is a simple $\mathbb{Q}W$-module constructed by Macdonald [3] via the reflection representation V of W.

The goal of this poster is to study the W-action on Φ_+^n and use it to understand the module $\mathcal{M}$.

An instructive example

The minimal-rank example occurs in type D_4, where we have:

- $\Phi_+ = \{\varepsilon_i \pm \varepsilon_j : 1 \leq i < j \leq 4\}$, $\Pi = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \varepsilon_3 - \varepsilon_4, \varepsilon_3 + \varepsilon_4\}$
- $\Phi_+^4 = \{\{\varepsilon_1 \pm \varepsilon_3, \varepsilon_2 \pm \varepsilon_4\}, \{\varepsilon_1 \pm \varepsilon_4, \varepsilon_2 \pm \varepsilon_3\}, \{\varepsilon_1 \pm \varepsilon_2, \varepsilon_3 \pm \varepsilon_4\}\}$
- The three 4-roots in Φ_+^4 form a partition of Φ_+, and we can identify them with the perfect matchings of $\{1, 2, 3, 4\}$ in an obvious way:



- If we denote the 4-roots listed above by C, N, and A from left to right, then C, N, and A contain 0, 1, and 3 simple roots, respectively, and in the module $\mathcal{M}$ we have the Ptolemy relation
  $$\gamma_C = \gamma_N + \gamma_A,$$
  which can also be viewed as a skein relation for the matchings.
- For any $\alpha = \varepsilon_i \pm \varepsilon_j \in \Phi_+$, the reflection s_α has the same effect as the transposition (i, j) on Φ_+^4, fixes the 4-root containing α, and interchanges the other two 4-roots in Φ_+^n. For example, if $\alpha = \varepsilon_1 + \varepsilon_3$, then s_α fixes C and interchanges N and A.

The above facts are crucial for studying general n-roots because, as we explain next, for general cases the changes a reflection can cause on an n-root will still occur in a D_4-subsystem.

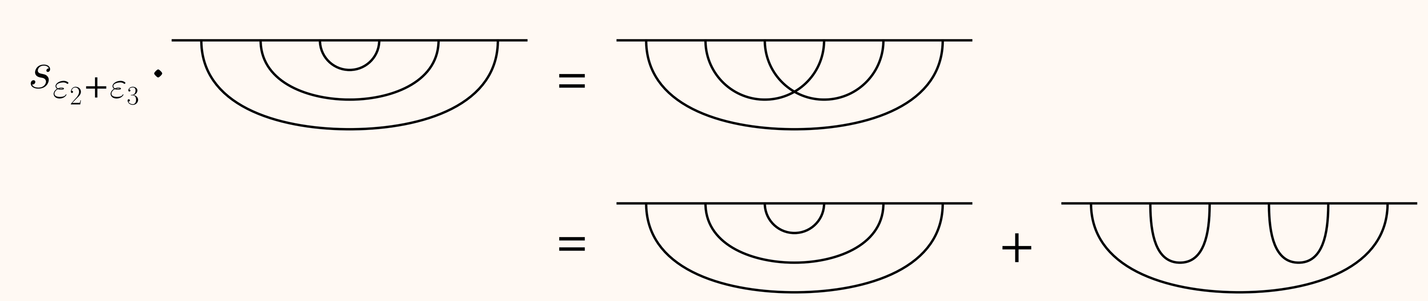
Actions of reflections

Let Φ be a root system of type E_7, E_8, or D_n for n even.

- We define a *coplanar quadruple in* Φ to be an orthogonal set $Q = \{\beta_1, \beta_2, \beta_3, \beta_4\}$ such that $\beta_1 + \beta_2 + \beta_3 + \beta_4 = 2\gamma$ for some root γ.
- If Q is a coplanar quadruple, then the set $\Psi_Q := \mathrm{Span}(Q) \cap \Phi$ can be shown to be a D_4-subsystem of Φ. Being a 4-root of Ψ_Q, Q must contain 0, 1, or 3 simple roots of Ψ_Q by the last section, in which cases we will call Q a *crossing*, *nesting*, or *alignment*, respectively.
- Coplanar quadruples control how reflections act on Φ_+^n:

Proposition. *Let $R \in \Phi_+^n$ and $\alpha \in \Phi_+$. If $\alpha \in R$, then $s_\alpha \cdot R = R$. If $\alpha \notin R$, then the set $Q := \{\beta \in R : \beta \not\perp \alpha\}$ is a coplanar quadruple such that $\alpha \in \Psi_Q$, and s_α acts on Q as explained in the last section (while fixing each element of $R \setminus Q$).*

Example. In type D_6, the following equations hold in $\mathcal{M}$:



As the reader may be suspecting, in type D_n for n even, $\mathcal{M}$ has a precise connection to the Specht module of the symmetric group $S_n \cong W(A_{n-1})$ indexed by the partition $(n/2, n/2)$. This is an appealing feature of the n-roots of this type.

Quasiparabolic structure

Definition (Rains–Vazirani [4]). Let (W, S) be a Coxeter system with set of reflections T. A *quasiparabolic set for W* is a pair (X, λ) where X is a W-set and λ is an integer-valued function such that

- $\forall s \in S, x \in X,\ |\lambda(sx) - \lambda(x)| \leq 1;$
- $\forall r \in T, x \in X,\ \lambda(rx) = \lambda(x) \implies rx = x;$
- $\forall s \in S, r \in T, x \in X,$
  $$\lambda(rx) > \lambda(x), \lambda(srx) < \lambda(sx) \implies rx = sx.$$

The *quasiparabolic order* on X is the weakest partial order $\leq_Q$ such that $x \leq_Q rx$ whenever we have $x \in X, r \in T$, and $\lambda(x) \leq \lambda(rx)$.

Theorem (Green–X. [2]) *Let Φ be a root system of type E_7, E_8, or D_n for n even. Define $\lambda : \Phi_+^n \to \mathbb{Z}$ by $\lambda(R) = c(R) + 2n(R)$, where c and n count the crossings and nestings in R, respectively. Then (Φ_+^n, λ) is a quasiparabolic set for the Weyl group W of Φ.*

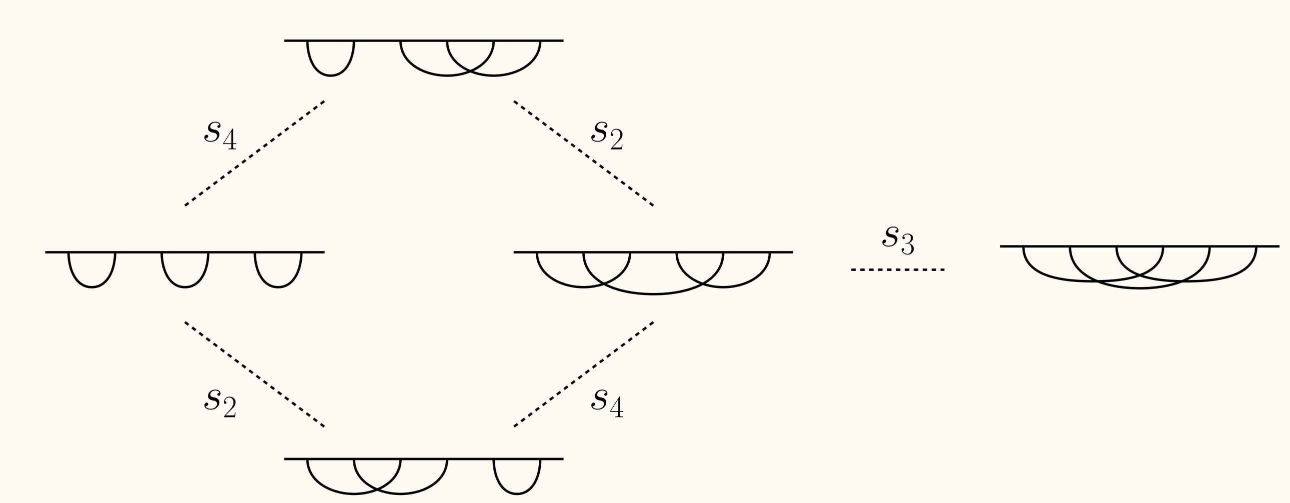
We note that for the type-D_n case the theorem can be deduced from known results about the fixed-point-free involutions of S_n, but our proof is type-independent.

Feature-avoiding elements

Theorem (Green–X. [2]) *Let Φ be a root system of type E_7, E_8, or D_n for n even. Let W be the Weyl group of Φ.*

1. *The set $B_{NC} = \{\gamma_R : R \in \Phi_+^n, R \text{ contains no crossing}\}$ is a basis of $\mathcal{M}$. Every element of Φ_+^n is a $\mathbb{Z}_+$-linear combination of B_{NC}; equivalently, with respect to B_{NC} every $w \in W$ acts on $\mathcal{M}$ by a matrix with sign-coherent columns.*
2. *The set $B_{NN} = \{\gamma_R : R \in \Phi_+^n, R \text{ contains no nesting}\}$ is also a basis of $\mathcal{M}$ and admits a unitriangular transition matrix to B_{NC} with nonnegative integer entries. It also has the structure of a distributive lattice induced by the left weak order $\leq_L$ on W.*
3. *The set $X_I = \{R \in \Phi_+^n : R \text{ contains no alignment}\}$ is a quasiparabolic set for a suitable parabolic subgroup W_I of W.*

Example. In type D_6, we have $B_{NN} \cong \{w \in W : w \leq_L s_2 s_4 s_3\}$:



Proposition. *The set $X_I \subseteq \Phi_+^n$ has the following properties:*

1. *If Φ has type D_n for $n = 2k$ even, then X_I is isomorphic as a poset to the symmetric group S_k under the strong Bruhat order.*
2. *If Φ has type E_7, then X_I admits a natural bijection to the 30 inequivalent labellings of the Fano plane.*
3. *If Φ has type E_8, then the following two graphs are not isomorhic but quantum isomorphic in the sense of Atserias et. al. [1]:*
   - *the graph where the vertices are the 120 even-level elements of X_I and where two 8-roots are adjacent if they are disjont;*
   - *the graph where the vertices are the 120 positive roots of Φ and where two roots are adjacent if they are orthogonal.*

References

[1] A. Atserias, L. Mančinska, D. E. Roberson, R. Šámal, S. Severini, and A. Varvitsiotis. Quantum and non-signalling graph isomorphisms. *Journal of Combinatorial Theory, Series B*, 136:289–328, 2019.

[2] R. M. Green and T. Xu. Orthogonal roots, Macdonald representations, and quasiparabolic sets. *Forum of Mathematics, Sigma*, 13:e110, 2025.

[3] I. G. Macdonald. Some irreducible representations of Weyl groups. *Bulletin of the London Mathematical Society*, 4(2):148–150, 1972.

[4] E. M. Rains and M. J. Vazirani. Deformations of permutation representations of Coxeter groups. *Journal of Algebraic Combinatorics*, 37(3):455–502, 2013.