Failure of the Lefschetz property for the Graphic Matroid

Ryo Takahashi   Mathematical Institute, Graduate School of Science, Tohoku University

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Setting

Artinian Gorenstein Algebra $A(f)$

- $f \in \mathbb{R}[x_1, \dots, x_n]$: homogeneous, degree d.
- $\mathrm{Ann}(f) \coloneqq \{\alpha \in \mathbb{R}[\partial_1, \dots, \partial_n] \mid \alpha f = 0\}$.
- $A(f) \coloneqq \mathbb{R}[\partial_1, \dots, \partial_n]/\mathrm{Ann}(f)$.
- Graded algebra: $A(f) = \bigoplus_{i=0}^{d} A_i$, with $\dim A_i = \dim A_{d-i}$.

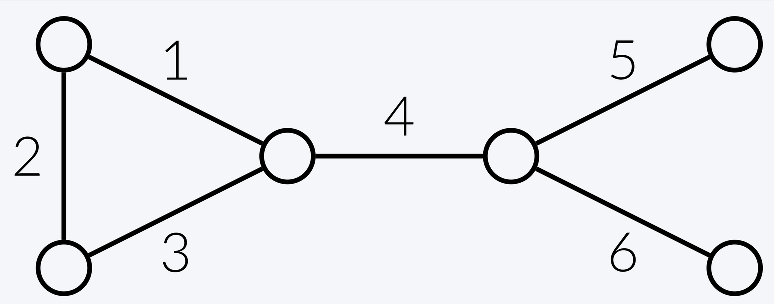
Strong Lefschetz Property (SLP)

- $A(f)$ has the **SLP** (in the narrow sense) if $\exists \ell \in A_1$ s.t. $\forall k \le d/2$, the map $\times l^{d-2k} : A_k \to A_{d-k}$ is an isomorphism.
- Equivalently: $\forall k \le d/2$, $\det \boldsymbol{H}_k \neq 0$, where
$$\boldsymbol{H}_k = ((\alpha_i \alpha_j) f)_{i,j=1}^{m} \in \mathbb{R}[x_1, \dots, x_n]^{m \times m}.$$
Here, $B_k = \{\alpha_1, \dots, \alpha_m\}$ is any basis of A_k.

Background

- The characterization of f for which $A(f)$ has the SLP is largely unknown.
- **Focus:** f as a matroid's **basis generating function**.
  Began with Maeno–Numata's work.
  Known: $\det \boldsymbol{H}_1 \neq 0$.
- We focus on graphs: For a graph G, the basis generating function is
$$f_G \coloneqq \sum_{T \text{ is a spanning tree of } G} \prod_{e \in E(T)} x_e.$$

Example



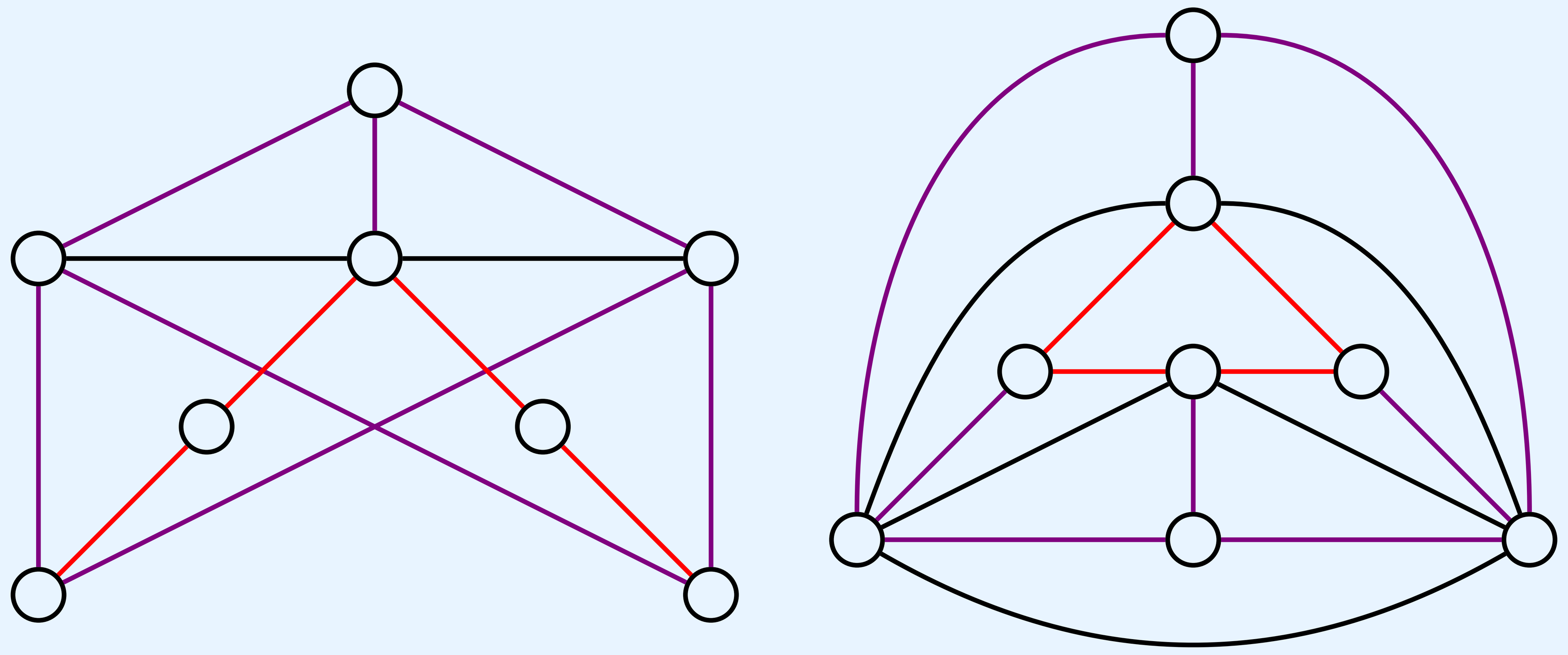
$f_G = (x_1x_2 + x_1x_3 + x_2x_3)x_4x_5x_6$.

Example graph G and its basis generating function f_G.

- $\mathrm{Ann}(f_G) = (\partial_1\partial_2 - \partial_1\partial_3,\ \partial_1\partial_2 - \partial_2\partial_3) + (\partial_i^2 \mid 1 \le i \le 6)$,
- $B_1 = \{\partial_i \mid 1 \le i \le 6\}$,
- $B_2 = \{\partial_i\partial_j \mid 1 \le i < j \le 6\} \setminus \{\partial_1\partial_3,\ \partial_2\partial_3\}$,
- $\det \boldsymbol{H}_1 \neq 0$, $\det \boldsymbol{H}_2 \neq 0$ by direct computation.

Graphs without the SLP

First examples G where $A(f_G)$ shows the failure of the SLP.



Graphs showing failure of the SLP. Coloring = degree in $\boldsymbol{F}$ (explained later).

Result: For these graphs, $\det \boldsymbol{H}_3 = 0$.

$\boldsymbol{H} \coloneqq \boldsymbol{H}_3$ is large:

- **Left:** 166×166 matrix, 13 variables.
- **Right:** 291×291 matrix, 17 variables.

Direct $\det \boldsymbol{H}$ computation is not possible.
How to confirm $\det \boldsymbol{H} = 0$? This is known as the *Edmonds' problem*.

Probabilistic Method

To determine $\det \boldsymbol{H} \neq 0$ probabilistically.

1. Choose random $(a_1, \dots, a_n) \in \mathbb{R}^n$.
2. Compute $\det(\boldsymbol{H}(a_1, \dots, a_n))$.
3. Non-zero $\Longrightarrow \det \boldsymbol{H} \neq 0$. Else: *likely* 0.

Very high probability of success with repeated trials.

Schwartz–Zippel Lemma

Let $g \neq 0$ be an n-variable polynomial and $S \subset \mathbb{R}$ be a finite subset. Then
$$\Pr[\, g(a_1, \dots, a_n) = 0 \mid (a_1, \dots, a_n) \in S^n \,] \le \frac{\deg g}{|S|}.$$

If $\det \boldsymbol{H} \neq 0$, random $(a_1, \dots, a_n)$ will not be a root of $\det \boldsymbol{H}$.

Deterministic Method

Verify $\det \boldsymbol{H} = 0$ without randomness.

Idea

Find non-zero vector of polynomials $\boldsymbol{F}$ s.t. $\boldsymbol{H}\boldsymbol{F} = \boldsymbol{0}$.

1. Collect $\boldsymbol{F}(a_1, \dots, a_n) \in \ker \boldsymbol{H}(a_1, \dots, a_n)$ at various $(a_1, \dots, a_n)$.
2. **Polynomial interpolation** for $\boldsymbol{F}$.

Observation

- The degree of x_i in $\boldsymbol{F}$ is computed: very small.
- Figure coloring = degree in $\boldsymbol{F}$ (black: 0, violet: 1, red: 2).
- This implies a feasible number of evaluation points are needed.

Challenges

This computation does not always succeed because

- $\boldsymbol{F}(a_1, \dots, a_n)$ is up to constant multiple, and
- $\ker \boldsymbol{H}(a_1, \dots, a_n)$ can be multi-dimensional.

Results

- **Probabilistic:** Found 152 graphs without the SLP (especially $\det \boldsymbol{H}_3 = 0$) out of $11,117$ simple connected graphs of 8 vertices.
- **Deterministic:** Definitively verified the failure of the SLP for 2 graphs.
- **Properties of $\boldsymbol{F}$:** Vector of degree 6 polynomials. This degree is remarkably small compared to the degree of column vectors of the adjugate matrix of $\boldsymbol{H}$.
  $\boldsymbol{F}$ identifies elements in the kernel of $\times l^{d-6}$: $A_3 \to A_{d-3}$.

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[3] S. Murai, T. Nagaoka, and A. Yazawa. Strictness of the log-concavity of generating polynomials of matroids. *J. Combin. Theory Ser. A*, 181:Paper No. 105351, 22, 2021.