When is the Chromatic Quasisymmetric Function Symmetric?

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Chromatic Quasisymmetric Function (CQF)

Let G be a graph on n vertices $1, 2, \ldots, n$. The chromatic quasisymmetric function of G is

$$X_G(\mathbf{x}; q) = X_G(x_1, x_2, \ldots; q) = \sum_{\substack{\kappa:[n]\to\mathbb{N} \\ \text{proper}}} q^{\mathrm{asc}(\kappa)} x_{\kappa(1)} x_{\kappa(2)} \cdots x_{\kappa(n)}$$

where κ is a proper coloring of G with colors from $\mathbb{N}$ and $\mathrm{asc}(\kappa)$ is the number of edges $\{i, j\}$ in G with $i < j$ and $\kappa(i) < \kappa(j)$ (such edges are called ascents of κ).

$$\begin{aligned} X_G(\mathbf{x}; q) &= (q^2 + 4q + 1)x_1x_2x_3 + q(x_1^2x_2 + x_1^2x_3 + x_2^2x_3 + x_1x_2^2 + x_1x_3^2 + x_2x_3^2) + \ldots \\ &= (q^2 + 4q + 1)M_{1,1,1} + qM_{2,1} + qM_{1,2} \\ &= (q^2 + 4q + 1)m_{1,1,1} + qm_{2,1} \qquad \text{(symmetric)} \end{aligned}$$

$$\begin{aligned} X_H(\mathbf{x}; q) &= (2q^2 + 2q + 2)x_1x_2x_3 + q^2(x_1^2x_2 + x_1^2x_3 + x_2^2x_3) + (x_1x_2^2 + x_1x_3^2 + x_2x_3^2) + \ldots \\ &= (2q^2 + 2q + 2)M_{1,1,1} + q^2M_{2,1} + M_{1,2} \qquad \text{(not symmetric)} \end{aligned}$$

A unit interval graph is a graph whose vertex set is a collection of unit length intervals on the real line and whose edges correspond to overlapping intervals. Shareshian and Wachs [6] showed that if G is a naturally labeled unit interval graph, then $X_G(\mathbf{x}; q)$ is symmetric and conjectured the following.

Conjecture (Shareshian, Wachs [6]):

The CQF of any naturally labeled unit interval graph is e-positive.

This conjecture generalizes the Stanley–Stembridge conjecture [7, 8] related to the immanants of Jacobi-Trudi matrices. Moreover, the polynomials $X_G(\mathbf{x}; q)$ for naturally labeled unit interval graphs are connected to the cohomology of regular semisimple Hessenberg varieties [2, 4].

Motivation/Approach

Only two families of graphs were previously known to have symmetric CQF's:

- cyclically labeled cycles (known to be e-positive [3])
- naturally labeled unit interval graphs (conjectured to be e-positive [6])

Motivating questions:

1. Is the CQF of a graph e-positive whenever it is symmetric?
2. When is the CQF of a graph symmetric?

Symmetric Products of Quasisymmetric Functions

Theorem:

Suppose $f \cdot g = h$ where f and g are quasisymmetric and h is symmetric. Then f and g are also symmetric.

Note: this does **NOT** hold when restricted to a finite number of variables (e.g. $x_1^2x_2 \cdot x_1x_2^2 = x_1^3x_2^3$)

Proof idea: Use the fact that the ring of symmetric functions is polynomially generated by the elementary symmetric functions and the ring of quasisymmetric functions is polynomially generated by a generating set that extends the set of elementary symmetric functions [5].

Given two graphs G and H, the CQF of their disjoint union $G \sqcup H$ satisfies

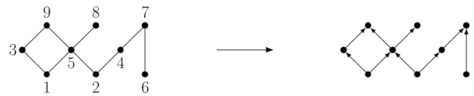
$$X_{G \sqcup H}(\mathbf{x}; q) = X_G(\mathbf{x}; q) X_H(\mathbf{x}; q).$$

Corollary:

The CQF of G is symmetric if and only if the CQF of every connected component is symmetric.

Necessary Conditions and the Classification of Symmetric Trees

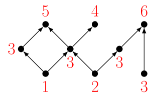
Every labeled graph G can be viewed as a directed acyclic graph and thus as a poset where $w < v$ if there is a directed path from w to v.



Proposition:

If $X_G(\mathbf{x}; q)$ is symmetric, then (a) the number of sources and sinks in G are equal and (b) the size of any antichain is at most the number of sources.

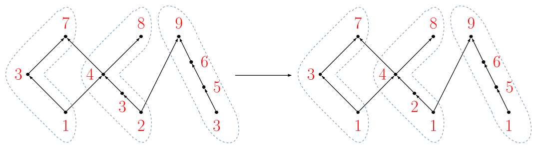
Proof idea: If not, there exists a proper coloring with weight $(1^t, s, 1^{n-s-t})$ having $|E|$ ascents, but not a proper coloring with weight $(s, 1^{n-s})$ and $|E|$ ascents where s is the size of the largest antichain and t is the number of vertices below this antichain.



Theorem:

Let G be connected. If $X_G(\mathbf{x}; q)$ is symmetric, then G has exactly one source and one sink.

Proof idea: If not, there exists a strictly injective map from proper colorings having weight $(1^k, a, 1^{n-k-a})$ and $|E|$ ascents to proper colorings having weight $(a, 1^{n-k-a})$ and $|E|$ ascents where a is the number of sources in G and k is some statistic.



Corollary:

If $X_G(\mathbf{x}; q)$ is symmetric, then G has the Hamiltonian path $1 - 2 - \cdots - n$ as a subgraph.

From this, we obtain a classification of all symmetric trees, answering an open question in [1].

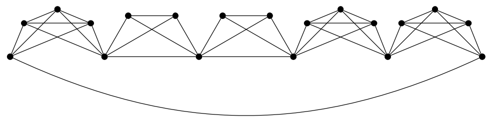
Corollary:

Let T be a tree. Then $X_T(\mathbf{x}; q)$ is symmetric if and only if T is a directed path.

Mixed Mountain Graphs

A (p, k, m)-mixed mountain graph is a string of m k-cliques (mountains) and $p - m$ $(k+1)$-cliques with one edge removed (bottomless moutains/cliques) such that:

1. each pair of adjacent (bottomless) mountains shares a single vertex
2. the end vertices of the first and last (bottomless) mountain are connected by an edge



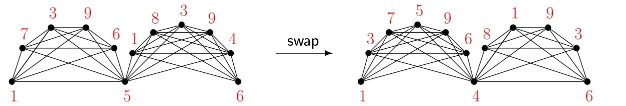
We label the vertices $1, 2, 3, \ldots, n$ from left to right. Note that cycles are the same as $(p, 2, p)$-mixed mountain graphs.

Theorem:

Let G and H both be (p, k, m)-mixed mountain graphs (we allow the positions of the k-mountains and $k+1$-bottomless mountains in G and H to differ). Then

$$X_G(\mathbf{x}; q) = X_H(\mathbf{x}; q).$$

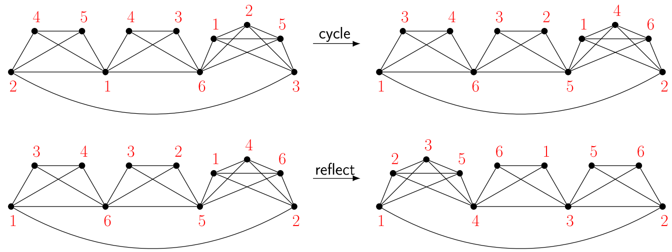
Proof idea: Find an ascent and weight-preserving bijective map that allows us to swap adjacent k-cliques and $k+1$-bottomless cliques.



Theorem:

Let G be a (p, k, m)-mixed mountain graph. Then $X_G(\mathbf{x}; q)$ is symmetric.

Proof idea: Find an ascent-preserving bijective map that swaps the number of occurences of the colors i and $i+1$. Our map makes use of the following ascent-preserving operations:



Open Questions/Future Work

1. Is the CQF of every (p, k, m)-mixed mountain graph e-positive? (true for all mixed mountain graphs with at most 10 vertices) For example, the e-expansion of the $(2, 3, 1)$-mixed mountain graph G (one 3-clique and one 4-bottomless clique) is given by:

$$\begin{aligned} X_G(\mathbf{x}; q) = {} & (q^7 + 3q^6 + 2q^5 + 2q^4 + 3q^3 + q^2)e_{3,3} + (q^7 + 2q^6 + q^5 + q^4 + 2q^3 + q^2)e_{4,2} \\ & + (q^8 + 3q^7 + 5q^6 + 11q^5 + 11q^4 + 5q^3 + 3q^2 + q)e_{5,1} \\ & + (q^9 + 4q^8 + 8q^7 + 17q^6 + 30q^5 + 30q^4 + 17q^3 + 8q^2 + 4q + 1)e_6 \end{aligned}$$

2. The only connected graphs with at most 8 vertices and a symmetric CQF are either naturally labeled unit interval graphs or mixed mountain graphs. Are these the only families of graphs with a symmetric CQF?

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