A BIJECTION BETWEEN TWO BRANCHING MODELS

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Abstract

We prove a bijection between the branching models of Kwon and Sundaram, very similar to the one conjectured by Lenart–Lecouvey in [1]. To do so, we use a symmetry of the Littlewood-Richardson coefficients in terms of the hive model. Along the way, we introduce a new branching rule using flagged hives.

Preliminaries - part 1

- $\mathfrak{sl}(2n,\mathbb{C}) = \{2n \times 2n \text{ traceless } \mathbb{C} \text{ matrices}\}$ is a Lie algebra with Lie bracket $[A,B] := AB - BA$.
- $\mathfrak{sp}(2n,\mathbb{C}) = \{A \in \mathfrak{sl}(2n,\mathbb{C}) \mid A^T S + SA = 0\}$ is a subalgebra. Here,

$$S = \begin{bmatrix} 0 & I_{n\times n} \\ -I_{n\times n} & 0 \end{bmatrix}$$

- A *partition* with at most m parts is a weakly decreasing m-tuple of non-negative integers.
- Partitions ν with at most $2n-1$ parts $\longleftrightarrow$ Finite dimensional simple modules of $\mathfrak{sl}(2n,\mathbb{C})$ (denoted $V(\nu)$).
- Partitions μ with at most n parts $\longleftrightarrow$ Finite dimensional simple modules of $\mathfrak{sp}(2n,\mathbb{C})$ (denoted $\tilde{V}(\mu)$).

Branching coefficients and models

Let $V(\nu)$ be a finite-dimensional simple module of $\mathfrak{sl}(2n,\mathbb{C})$. Consider its restriction to $\mathfrak{sp}(2n,\mathbb{C})$:

$$\mathrm{Res}^{\mathfrak{sl}(2n,\mathbb{C})}_{\mathfrak{sp}(2n,\mathbb{C})} V(\nu) = \bigoplus_{\mu} \tilde{V}(\mu)^{\oplus c^{\nu}_{\mu}}$$

- The multiplicities c^{ν}_{μ} are called *branching coefficients*.
- By a *branching model*, we mean a combinatorial set whose cardinality equals the branching coefficient. i.e., it is a rule associating a combinatorial set A, to any given pair (ν,μ), satisfying $\mathrm{card}(A) = c^{\nu}_{\mu}$

Preliminaries - part 2

- Let $[m] := \{1,2,\ldots,m\}$. For a word w in $[m]$, its weight is $(a_1,a_2,\ldots)$ where a_i is the multiplicity of i in w.
- A word is *dominant* if the weight of every prefix of it is a partition.
- Given a partition $\nu = (\nu_1,\nu_2,\ldots,\nu_m)$, the (Young) diagram of ν is the left and top justified collection of boxes such that given any i, the number of boxes in i-th row is ν_i.
- Given two partitions ν, μ such that $\mu \subset \nu$ (i.e., $\mu_i \leq \nu_i$ for all i), the (skew) shape ν/μ is the collection of boxes in the diagram of ν that do not belong to the diagram of μ. We identify the shape $\nu/(0)$ with ν.
- A (semistandard) tableau of shape ν/μ is a filling of the boxes in ν/μ by positive integers such that the rows are weakly increasing and the columns are strictly increasing.
- Reversed row word of a tableau is the word obtained by reading the rows of the tableau right to left starting with the top row and proceeding downwards. The weight of a tableau is the weight of its reversed row word.
- $\mathrm{Tab}(\nu/\mu,\lambda)$ is the set of all tableaux of shape ν/μ and weight λ.
- $LR(\nu/\mu,\lambda)$ is all tableaux $T \in \mathrm{Tab}(\nu/\mu,\lambda)$ whose reversed row word is dominant.
- $T \in \mathrm{Tab}(\nu/\mu)$ is λ dominant if the reversed reading word of T post-adjoined to the word $1^{\lambda_1}2^{\lambda_2}\ldots$ is dominant
- $\mathrm{LR}^{\nu}_{\lambda,\mu} := \{T \in \mathrm{Tab}(\mu) | T \text{ is } \lambda\text{-dominant and } wt(T) = \nu - \lambda\}$

Example:

$T =$

		1	1
	1	2	
	2		
2			

reversed row word$(T) = 112122$; weight $= (3,3,0,0,0,\cdots)$

Sundaram's branching model

Definition 1. *A tableau T belongs to $\mathrm{LRS}(\nu/\mu,\lambda)$ if*

S1: $T \in \mathrm{LR}(\nu/\mu,\lambda)$

S2: If the integer $2i+1$ appeared in row r of T, then $r \leq n+i$

Theorem 1. *[2] Let ν be a partition with at most $2n-1$ parts and μ be a partition with at most n parts. Then,*

$$c^{\nu}_{\mu} = \textit{cardinality}\left(\bigcup \mathrm{LRS}(\nu/\mu,\lambda)\right)$$

where, the union is over all even partitions λ, i.e., those λ for which $\lambda_{2i-1} = \lambda_{2i}$ for each positive integer i.

Kwon's branching model

Definition 2. *A tableau T belongs to $\mathrm{LRK}^{\nu}_{\lambda,\mu}$ if*

K1: $T \in \mathrm{LR}^{\nu}_{\lambda,\mu}$

K2: For each positive integer i, entries of $S(T)$ in row i are at least $2i-1$.

where, S denotes the Schützenberger Involution.

Theorem 2. *[1, 3]*

$$c^{\nu}_{\mu} = \textit{cardinality}\left(\bigcup \mathrm{LRK}^{\nu}_{\lambda,\mu}\right)$$

here, the union is over all even partitions λ.

Conjecture of Lenart-Lecouvey

Proposition 1. *The companion map denoted by c, which records row numbers of the entries as a tableau, is a bijection from $\mathrm{LR}(\nu/\mu,\lambda)$ to $\mathrm{LR}^{\nu}_{\mu,\lambda}$.*

Example:

$T =$

		1	1
	1	2	
	2		
2			

$\mathrm{c}(T) =$

1	1	2
2	3	4

Reversed row word of $c(T)$ post-adjoined to 1123 ($\mu = (2,1,1)$) is 1123211432, which clearly is dominant.

Note that

1. $\mathrm{LRS}(\nu/\mu,\lambda) \hookrightarrow \mathrm{LR}^{\nu}_{\mu,\lambda}$ through the companion map
2. $\mathrm{LRK}^{\nu}_{\lambda,\mu} \hookrightarrow \mathrm{LR}^{\nu}_{\lambda,\mu}$ by definition

Therefore a bijection between $\mathrm{LR}^{\nu}_{\mu,\lambda}$ and $\mathrm{LR}^{\nu}_{\lambda,\mu}$ might in principle give a bijection between $\mathrm{LRS}(\nu/\mu,\lambda)$ and $\mathrm{LRK}^{\nu}_{\lambda,\mu}$.

Conjecture 1 (Lenart-Lecouvey)**.** *The bijection of Hendriques-Kamnitzer between $\mathrm{LR}^{\nu}_{\mu,\lambda}$ and $\mathrm{LR}^{\nu}_{\lambda,\mu}$ restricts to a bijection between $\mathrm{LRS}(\nu/\mu,\lambda)$ and $\mathrm{LRK}^{\nu}_{\lambda,\mu}$.*

Main result

There are several bijections between $\mathrm{LR}^{\nu}_{\mu,\lambda}$ and $\mathrm{LR}^{\nu}_{\lambda,\mu}$ known in the literature including but not restricted to

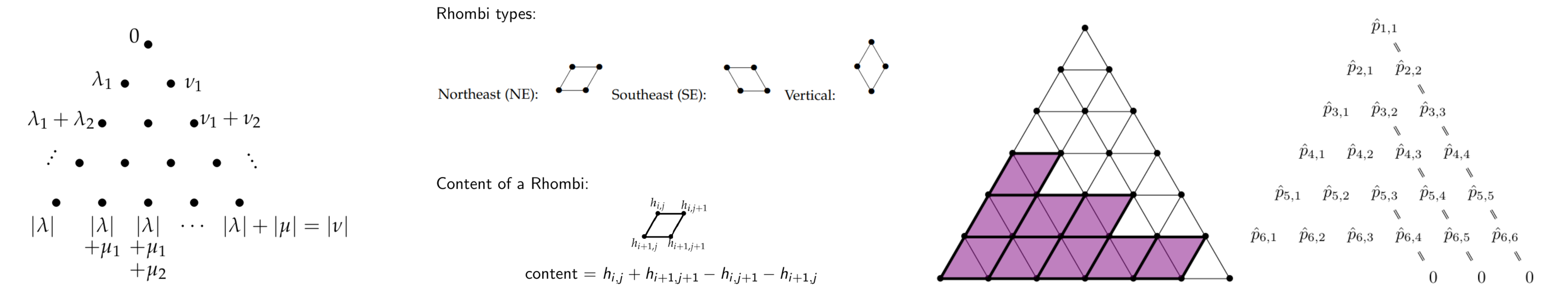
1. Hendriques–Kamnitzer
2. Azenhas–King–Terada
3. Kushwaha–Raghavan–Viswanath

Theorem 3 (Sathish-Torres)**.** *The bijection of Kushwaha–Raghavan–Viswanath between $\mathrm{LR}^{\nu}_{\mu,\lambda}$ and $\mathrm{LR}^{\nu}_{\lambda,\mu}$ restricts to a bijection between $\mathrm{LRS}(\nu/\mu,\lambda)$ and $\mathrm{LRK}^{\nu}_{\lambda,\mu}$.*

The proof is done by careful unwinding of the bijection of Kushwaha–Raghavan–Viswanath.

Hives and branching

An element of $\mathrm{Hive}_{\mathbb{Z}}(\lambda,\mu,\nu)$ (called an integral hive) is a triangular array of integers satisfying boundary constraints as in the picture below, and that all rhombi contents are non-negative.



Theorem 4. *[4] There is a bijection (denoted φ) between $LR^{\nu}_{\lambda,\mu}$ and the set of integral hives $\mathrm{Hive}_{\mathbb{Z}}(\lambda,\mu,\nu)$*

Corollary 1. *The bijection φ when pre-composed with the companion map restricts to a bijection between $\mathrm{LRS}(\nu/\mu,\lambda)$ and the set of integral flagged hives $\mathrm{Hive}_{\mathbb{Z}}(\mu,\lambda,\nu,\phi)$ where $\phi = (n, n+1, n+1, n+2, n+2, \ldots)$.*

For $n = 3$, flagged hives are schematically expressed in the diagram above where the shaded rhombi are constrained to have content zero.

The bijection of Kushwaha–Raghavan–Viswanath

Let λ be a partition with atmost m parts. A Gelfand–Tsetlin pattern P of shape λ is a triangular array of numbers $(p_{i,j})_{1\leq i\leq m, 1\leq j\leq i}$ such that

$$p_{i+1,j} \geq p_{ij} \geq p_{i+1,j+1} \qquad \& \qquad p_{m,j} = \lambda_j \tag{1}$$

for all $1 \leq i < m, 1 \leq j \leq i$.

- Given a partition λ there is a bijection GT from the set $\mathrm{Tab}(\lambda)$ to the set of all Gelfand–Tsetlin patterns of shape λ.
- There are injective maps P and $\hat{P}$ from $\mathrm{Hive}(\lambda,\mu,nu)$ to the set of Gelfand–Tsetlin patterns of shape μ and λ respectively.
- There are bijective maps T and C from the set of all Gelfand–Tsetlin patterns of shape λ to $\mathrm{Tab}(\lambda)$ and $\mathrm{Tab}(\bar{k}/(\bar{k} - \mathrm{rev}(\lambda)))$ respectively. Here, $\bar{k}$ denotes the partition $(k,k,\ldots,k)$ and rev is the function that reverses a tuple.

The bijection of Kushwaha–Raghavan–Viswanath is

$$\mathrm{rect} \circ C \circ \hat{P} \circ \varphi$$

Here, the map rect is the rectification operation of skew semistandard tableaux to tableaux.

Proposition 2 (Sathish-Torres)**.** *The set $\mathrm{LRS}(\nu/\mu,\lambda)$ under the bijection of Kushwaha–Raghavan–Viswanath precomposed with the companion map c, maps inside (and hence maps bijectively onto) $\mathrm{LRK}^{\nu}_{\lambda,\mu}$. Therefore,*

$$c^{\nu}_{\mu} = \textit{cardinality}\Big(\bigcup_{\lambda} \mathrm{Hive}_{\mathbb{Z}}(\mu,\lambda,\nu,\phi)\Big)$$

where, the union runs over all even partitions λ.

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