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Asymptotic count of edge-bicolored graphs

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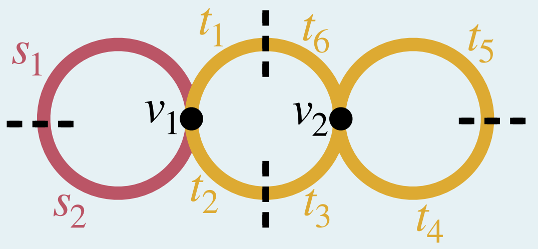
1. Edge-bicolored graphs

Given two disjoint finite sets S and T of labels, an $[S,T]$**-labeled edge-bicolored graph** is a tuple (V, E_S, E_T), where

- the vertex set V is a partition of $S \cup T$,
- E_S is a partition of S into blocks of size 2,
- E_T is a partition of T into blocks of size 2.



Many ways to assign labels to the same graph

$|\mathrm{Aut}(G)| = 8$

$\mathscr{G}$ = set of isomorphism classes of graphs

2. Generating function

The generating function for graphs with given bidegrees is

$$\sum_{G\in\mathscr{G}} \frac{\eta^{|E|}}{|\mathrm{Aut}(G)|} \prod_{v\in V} \lambda_{\deg(v)} = \sum_{s,t\geq 0} \eta^{s+t}\cdot(2s-1)!!\cdot(2t-1)!!\cdot a_{2s,2t}(\lambda)$$

where

$$\sum_{s,t\geq 0} a_{s,t}(\lambda)\, x^s y^t = \exp\left(\sum_{\substack{u,w\geq 0\\ u+w\geq 1}} \lambda_{u,w} \frac{x^u y^w}{u!w!}\right)$$

3. Exponential integral

$$I(z) = \frac{z}{2\pi}\int_D \exp\left(z\, g(x,y)\right)\, \mathrm{d}x\,\mathrm{d}y$$

marginal likelihood integral (Bayesian statistics)

Zero-dimensional path integral (Quantum field theory)

4. Connection

Theorem: Assume $g(x,y) = -\frac{x^2}{2} - \frac{y^2}{2} + \sum_{u+w\geq 3} \Lambda_{u,w} \frac{x^u y^w}{u!w!}$, then for large z,

(variant of Wick's theorem)

$$I(z) \sim \sum_{n\geq 0} \underbrace{\left(\sum_{G\in\mathscr{G}^\star_{-n}} \frac{1}{|\mathrm{Aut}(G)|}\prod_{v\in V}\Lambda_{\deg(v)}\right)}_{A_n} z^{-n}$$

where $\mathscr{G}^\star_{-n}$ is the set of all edge-bicolored graphs with vertex degrees at least 3 and Euler characteristic $-n$.

The non-zero monomials in g determine the graphs involved in the expression for A_n.

5. Asymptotics

Let us restrict to k-regular graphs, i.e., restrict to $g(x,y)$ of the form

$$g(x,y) = -\frac{x^2}{2} - \frac{y^2}{2} + \sum_{u+w=k} \Lambda_{u,w} \frac{x^u y^w}{u!w!}$$

NEW!

Theorem: $A_n \sim \frac{1}{2\pi}\Gamma(n) \sum_{(x,y)\in\Psi} \frac{(-g(x,y))^{-n}}{\sqrt{-\det H_g(x,y)}}, \quad n\to\infty,$

where Ψ is the set of non-degenerate critical points of g in $\mathbb{C}\cdot\mathbb{R}^2$ closest to the origin, and H_g is the Hessian matrix.

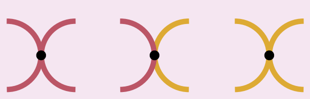
Phase transitions: $\frac{1}{3}$, 3

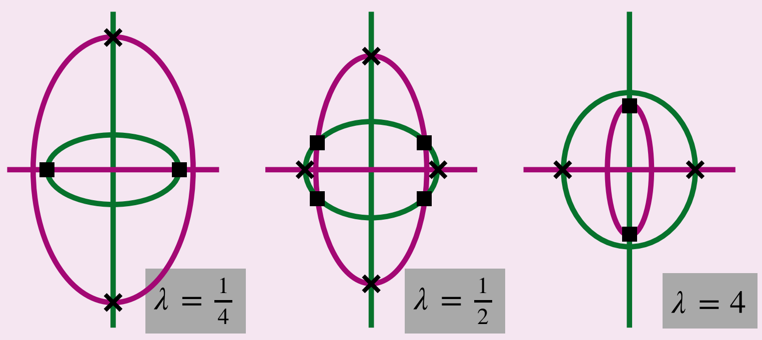
	Ψ	$A_n \sim$
$0<\lambda<\frac{1}{3}$	$\left(\pm\sqrt{6},0\right)$	$\frac{\Gamma(n)}{\pi}\sqrt{\frac{1}{2-6\lambda}}\left(\frac{2}{3}\right)^n$
$\frac{1}{3}<\lambda<3$	$\left(\pm\sqrt{\frac{9-3\lambda}{4\lambda}},\pm\frac{\sqrt{9\lambda-3}}{2\lambda}\right)$	$\frac{\Gamma(n)}{\pi}\sqrt{\frac{8\lambda}{-3\lambda^2+10\lambda-3}}\left(\frac{-16\lambda^2}{3\lambda^2-18\lambda+3}\right)^n$
$\lambda>3$	$\left(0,\pm\frac{\sqrt{6}}{\lambda}\right)$	$\frac{\Gamma(n)}{\pi}\sqrt{\frac{\lambda}{2\lambda-6}}\left(\frac{2\lambda^2}{3}\right)^n$

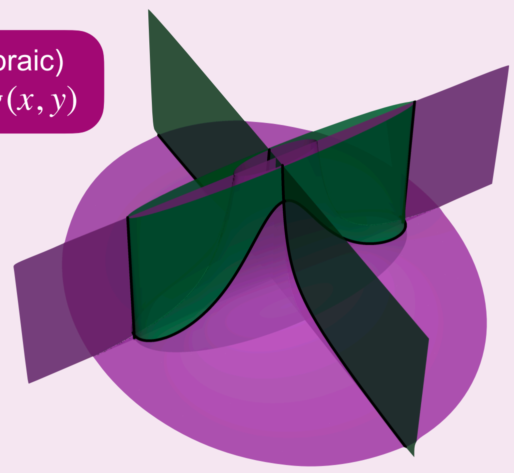
Corresponding to (algebraic) geometric properties of $g(x,y)$

6. Critical Ising model

$$g(x,y) = -\frac{x^2}{2} - \frac{y^2}{2} + \frac{x^4}{4!} + \lambda\frac{x^2y^2}{2!2!} + \lambda^2\frac{y^4}{4!}$$







λ

$\frac{\partial g}{\partial x}(x,y) = 0$

$\frac{\partial g}{\partial y}(x,y) = 0$

References

"***Bivariate exponential integrals and edge-bicolored graphs***" M. Borinsky, C. Meroni, M. Wiesmann, Le Matematiche, 80 (1), 167-187 (2025)