Relations Between Generalised Gelfand-Tsetlin and Kazhdan-Lusztig Bases of S_n

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Abstract

We prove that the Kazhdan-Lusztig basis of Specht modules is upper triangular with respect to all generalized Gelfand-Tsetlin bases constructed from any multiplicity free tower of standard parabolic subgroups.

Background: Specht Modules and Standard Young Tableaux

Notation: $\mathrm{SYT}(\lambda) :=$ Standard Young Tableaux of shape λ.

For a given Specht module, V^λ, $\lambda \vdash n$ (an irreducible representaion of S_n) there are two notable bases indexed by $\mathrm{SYT}(\lambda)$

- **Kazhdan-Lusztig Basis**: A unique basis constructed using the Hecke algebra. This is a **canonical** basis. We denote this basis as $\{c_T \mid T \in \mathrm{SYT}(\lambda)\}$
- **Gelfand-Tsetlin Basis**: A basis constructed by decomposing a Specht module along a multiplicity free chain of subgroups. Traditionally, the chain

$$S_2 < S_3 < \cdots < S_{n-1} < S_n$$

is commonly used, we call this the *standard chain*.

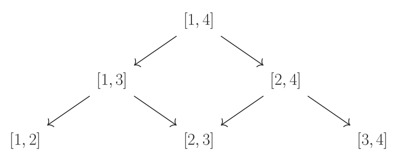
The Generalised Gelfand-Tsetlin Basis

The **generalised Gelfand-Tsetlin bases** of a Specht Module are constructed by decomposing along a multiplicity free chain of parabolic subroups of S_n.

$$G_{n-1} < G_{n-2} < \cdots < G_2 < G_1 = S_n$$

Where $G_k = S_{[a,b]}$ and $a - b + n = k$.

Consider the 4 multiplicity free chains of S_4, we denote $[a,b] := S_{[a,b]}$ for convenience.



The Motivating Problem

We have two important discoveries:

- Garsia-MacLarnan (1988): The Kazhdan-Lusztig basis is upper triangular with Young's Natural basis
- Armon-Halverson (2021): Young's Natural basis is upper triangular with Young's Seminormal basis (Gelfand-Tsetlin basis).

These two results imply that the Kazhdan-Lusztig basis is upper triangular with respect to the Gelfand-Tsetlin basis. This begs the question:

> As the construction of the Kazhdan-Lusztig basis does not depend on any choices, whereas the Gelfand-Tsetlin basis does, is the Kazhdan-Lusztig basis upper triangular with respect to all generalised Gelfand-Tsetlin bases?

Not A Trivial Task

These generalised Gelfand-Tsetlin bases are **not** upper triangular with respect to each other.

A Quick Example

Let $\lambda = (2,1) \vdash 3$, we will take $V^\lambda \subset \mathbb{C}^3$ to be the space of vectors whose entries sum to 0, i.e the 2-dimensional standard representation.

Consider the two Gelfand-Tsetlin bases $\mathbf{B}_1$ and $\mathbf{B}_2$ constructed through the chains $([1,2],[1,3])$ and $([2,3],[1.3])$ respectively

- $\mathbf{B}_1 = \{(1,1,-2),(1,-1,0)\}$
- $\mathbf{B}_2 = \{(-2,1,1),(0,-1,1)\}$

$\mathbf{B}_1$ and $\mathbf{B}_2$ do not share a common vector (upto scalar multiplation), hence they cannot be upper triangular with respect to each other.

Main Result

Theorem 1: The Kazhdan-Lusztig basis of a Specht module is upper triangular with respect to all generalised Gelfand-Tsetlin bases constructed through any multiplicity free chain of standard parabolic subgroups.

Notation Section

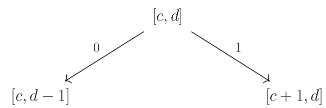
Notation on SYT(λ)

Let $\lambda \vdash n$ and $T \in \mathrm{SYT}(\lambda)$. Let

- $d(T)$ be the tableau obtained after removing the n-box from T.
- If $T = d(S)$, then we denote $T^+ := S$
- $\overline{T} := \mathrm{ev}(T)$, where $\mathrm{ev} : SYT(\lambda) \to \mathrm{SYT}(\lambda)$ is Schutzenberger's evacuation operator
- $\varepsilon_i(T) :=$ is the row number of the i-box of T.

Labelling Chains As Binary Sequences

We use binary sequences to label chains. If the largest element is removed from the parent group, we label that step with 0. Similarly, for the smallest element we label that step with 1.



We call the generalised Gelfand-Tsetlin basis constructed along a chain b, a b-GT basis, and we denote it as $\{v_T^b \mid T \in \mathrm{SYT}(\lambda)\}$

For $b = b_1 \cdots b_n \in \mathbb{B}_n$, we set

$$\overline{b} = \overline{b_1} \cdots \overline{b_n}$$
$$b_\dagger = b_2 \cdots b_n$$

where $\overline{0} = 1$ and vice versa.

A Bijection and Ordering for Each Chain

Let $b \in \mathbb{B}_{n-2}$

1. Define a bijection $\varphi_b = \varphi_{\lambda,b} \in \mathrm{Aut}(\mathrm{SYT}(\lambda))$ recursively:
$$\varphi_{\lambda,b}(T) = \begin{cases} \varphi_{\mu,b_\dagger}(d(T))^+ & \text{if } b_1 = 0, \\ \varphi_{\mu,\overline{b_\dagger}}(d(\overline{T}))^+ & \text{if } b_1 = 1, \end{cases}$$
where $\mu = \mathrm{sh}(d(T))$ or $\mathrm{sh}(d(\overline{T}))$.
2. Define a linear order $\leq_b = \leq_{\lambda,b}$ recursively
$$S \leq_{\lambda,b} T \Leftrightarrow \begin{cases} \varepsilon_n(S) < \varepsilon_n(T) \text{ or } \varepsilon_n(S) = \varepsilon_n(T) \text{ and } d(S) \leq_{\mu,b_\dagger} d(T) \text{ if } b_1 = 0, \\ \varepsilon_n(\overline{S}) < \varepsilon_n(\overline{T}) \text{ or } \varepsilon_n(\overline{S}) = \varepsilon_n(\overline{T}) \text{ and } d(\overline{S}) \leq_{\mu,\overline{b_\dagger}} d(\overline{T}) \text{ if } b_1 = 1, \end{cases}$$
where $\mu = \mathrm{sh}(d(T))$ or $\mathrm{sh}(d(\overline{T}))$.

Non-Trivial Examples

The order can be described by associating a sequence of numbers $\mathrm{sig}_b(T)$ to a tableau T, and ordering the sequences reverse lexicographically.

$\mathbf{b}$	$\varphi_\mathbf{b}(\mathbf{T})$	$\mathrm{sig}_\mathbf{b}(\mathbf{T})$
$00\cdots0$	id	$(\varepsilon_1(T),\ldots,\varepsilon_n(T))$
$11\cdots1$	$\overline{T}$	$(\varepsilon_1(\overline{T}),\ldots,\varepsilon_n(\overline{T}))$
$01\cdots1$	$(\overline{d(T)})^+$	$(\varepsilon_1(\overline{T}),\ldots,\varepsilon_{n-1}(\overline{T}),\varepsilon_n(T))$

Main Result (More Detail)

Let $\lambda \vdash n$ and $b \in \mathbb{B}_{n-2}$. Consider a Specht module V^λ with a Kazhdan-Lusztig basis $\{c_T \mid T \in \mathrm{SYT}(\lambda)\}$ and a b-Gelfand-Tsetlin basis, $\{v_T^b \mid T \in \mathrm{SYT}(\lambda)\}$.

We have that for any $T \in \mathrm{SYT}(\lambda)$,

$$c_T = a_T v^b_{\varphi_b(T)} + \sum_{S <_b T} a_S v^b_{\varphi_b(S)}, \tag{1}$$

where $a_T \neq 0$. Moreover, the b-GT basis can be normalised so that the coefficients $a_T, a_S \in \mathbb{Z}$.

A Critical Lemma

For$\lambda \vdash n$, let

$$V^\lambda_{(i)} = \mathrm{span}\{c_T \mid \varepsilon_n(T) \leq i\}.$$

If λ has r removable boxes we have the filtration

$$0 \subset V^\lambda_{(a_1)} \subset V^\lambda_{(a_2)} \cdots \subset V^\lambda_{(a_r)} = V^\lambda.$$

> (Gassow-Yacobi, 2022): Let $\mu_i \vdash n-1$, be the partition after removing the n-box from the i-th row of T and let $d(T) \in \mathrm{SYT}(\mu_i)$. The map:
> $$f_i : V^\lambda_{a_i} / V^\lambda_{a_{i-1}} \xrightarrow{\sim} V_{\mu_i}$$
> $$c_T + V_{a_{i-1}} \longmapsto c_{d(T)}$$
> is an isomorphism.

We show that this isomorphism acts similarly to b-Gelfand-Tsetlin basis vectors. This allows us to induct the relationship from tableaux of size k to $k+1$.

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