On z-Superstable and Critical Configurations of Chip Firing Pairs

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About Our Project

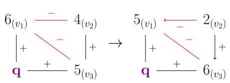
Chip firing is a game played on graphs where 'chips' are placed on each vertex and distributed across the graph through 'firings'. This game simulates exchange between entities and has applications in fields like biology, physics, and even business communications. We study chip firing on **signed graphs**—that is, graphs with positively or negatively signed edges.

Chip Firing Basics

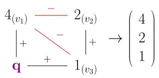
Imagine placing some number of poker chips on each vertex of a signed graph. Label one of the vertices q: this is the *sink* vertex, and it can be thought of as having unlimited chips. To **fire** a vertex, we do the following to each of its neighbors:

- Over a positive edge, move a chip from the fired vertex to its neighbor.
- Over a negative edge, remove one chip from both vertices.

For example, here we fire vertex 2:



We represent the state of the game as a **configuration**, which is an integral vector that encodes the number of chips on each vertex. Note that we do not include the sink q in configurations, since we do not put chips on it.



We define **valid signed configurations** on a signed graph with respect to the all-positive graph G_+: Let M be the Laplacian of G_+, and L be the Laplacian of G_ϕ. Then, $\vec{s}$ is valid only when $ML^{-1}\vec{s}$ has no negative entries.

The set of all valid signed configurations is called S^+. Note that we frequently use a different set of valid configurations, $R^+ = \{ML^{-1}\vec{s} : \vec{s} \in S^+\}$, to analyze the relationships between multiple signed graphs.

Our Goals

Research on signed chip firing began in 2022 with [1]. Our research this summer builds on their work, finding answers to the two main questions left to us from their paper:

- **Vertex switching**: Graphs that are *switching equivalent* always have the same critical group structure. What is the relationship between the superstable configurations of these graphs? (***Theorem 1***)
- **Duality**: There is a natural bijection between superstable and critical configurations for unsigned graphs, but no such duality has been found for signed graphs until now. In our research, we constructed a bijection between superstable and critical configurations for signed graphs that naturally extends the unsigned duality. (***Theorem 2***)

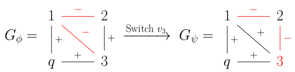


Figure 1: Two switching equivalent signed graphs.

Special Configurations and Critical Groups

A **superstable configuration** is one where firing any set of vertices results in an invalid configuration.
A **critical configuration** is one that is both **stable** and **recurrent**.

- A *stable* configuration is one such that firing any vertex results in an invalid configuration.
- A *recurrent* configuration is one where, after firing the sink q, it is possible to return to the same configuration after some number of valid non-sink firings.

Given a signed graph G_ϕ, we denote the set of superstable and critical configurations in R^+ as $\mathrm{sstab}(G_\phi)$ and $\mathrm{crit}(G_\phi)$ respectively. For unsigned graphs, there is a duality between superstable and critical configurations:

$$\mathrm{flip} : \mathrm{sstab}(G) \to \mathrm{crit}(G),\ \mathrm{flip}(\vec{s}) = \vec{c}_{max} - \vec{s} \text{ where } \vec{c}_{\max} \text{ is the configuration with deg } -1 \text{ chips at each vertex.}$$

Configurations $\vec{c}$ and $\vec{d}$ are **firing-equivalent** ($\vec{c} \sim_L \vec{d}$) if there is some sequence of firings that transforms $\vec{c}$ into $\vec{d}$. Alternatively, $\vec{c}$ and $\vec{d}$ are firing-equivalent when $L^{-1}(\vec{c} - \vec{d})$ is an integer vector. Each equivalence class under $\sim_L$ contains exactly one superstable configuration and exactly one critical configuration.

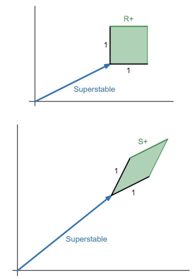
We can make this set of equivalence classes a group by giving it a group operation. A natural operation is simply adding the configurations termwise. The group is denoted $\mathcal{K}(G)$.

Computing Special Configurations

Although verifying that a configuration is superstable is easy, computing all of the superstable configurations of some G_ϕ is a challenge. To find them, we must compare G_ϕ to its unsigned graph G_+. Here is the process:

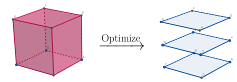
1. Find all the superstable vectors of G_+. This can be done easily by chip firing until you can't anymore.
2. Mark a unit square on the tip of each superstable vector. Fittingly, these squares are denoted $\square_{\vec{v}}$.
3. Transform the marked spaces by LM^1, and find the integral points in each parallelogram. Any integral points found are superstable configurations in S^+.

Note that, while squares are used in this example, the dimension of the search space is the number of non-sink vertices in the graph. We would actually be searching $n-1$ dimensional polyhedra.



While this method works, it is slow. Finding the integral points in a polyhedron becomes exponentially harder with added dimensions, so finding a way to reduce the dimension will improve efficiency by orders of magnitude.

If a vertex has only positive incident edges (that is, it is *locally positive*), then the entry corresponding to that vertex will always be integral in the preimage of any signed configuration. This allows us to substantially reduce the time it takes to compute superstables for graphs with locally positive vertices.



Vertex Switching Isomorphism

Vertex switching is an operation on signed graphs. To switch a vertex v, invert the sign of every edge incident to v. Two graphs G_ϕ and G_ψ are **switching equivalent** if some sequence of vertex switches transforms G_ϕ into G_ψ (***Figure 1***). Another definition is if the Laplacians L_ϕ and L_ψ satisfy $L_\phi = \hat{E}L_\psi\hat{E}$ for some diagonal matrix $\hat{E}$ whose v-th diagonal entry is -1 if vertex v is flipped, and 1 otherwise.

Theorem 1: Switching Isomorphism

Theorem 1. *Given two switching-equivalent graphs G_ϕ and G_ψ such that $L_\phi = \hat{E}L_\psi\hat{E}$, the map*

$$\lambda : v \mapsto \hat{E}v$$

is an isomorphism between $\mathcal{K}(G_\phi)$ and $\mathcal{K}(G_\psi)$.

This isomorphism allows us to compute special configurations faster. Given a signed graph, find some switching equivalent graph with a maximal number of locally positive vertices. On this new graph we apply the dimension reduction described in "***Computing Special Configurations***" to compute superstables of that switching equivalent graph. Finally, we use the vertex switching isomorphism to convert those superstables back into superstables of the original graph.

Signed Duality

We can construct a duality between signed superstable and critical configurations. First, we will define the map μ that is an involution on the set of superstable configurations of the underlying unsigned graph G_+:

$$\mu(\vec{s}) = \begin{cases} \vec{s} & \text{if } \{LM^{-1}2\vec{s}\} = \{LM^{-1}\vec{c}_{\max}\}, \\ \mathrm{sstab}(\vec{c}_{\max} - \vec{s}) & \text{otherwise} \end{cases}$$

where $\mathrm{sstab}(\vec{c}_{\max} - \vec{s})$ refers to the unique superstable configuration that is firing-equivalent to $\vec{c}_{\max} - \vec{s}$. Then, we can define the duality sflip that maps signed superstable configurations to signed critical configurations in R^+.

Theorem 2: Signed Duality

Theorem 2. *Given a signed graph G_ϕ, the map $\mathrm{sflip} : \mathrm{sstab}(G_\phi) \to \mathrm{crit}(G_\phi)$ given by*

$$\vec{s} \mapsto \vec{c}_{max} - \mu(\lfloor \vec{s} \rfloor) + \{\vec{s}\}$$

bijects the superstable and critical configurations of G_ϕ.

Not only is sflip a bijection between superstable and critical configurations in R^+, it also recovers the usual unsigned duality (the flip map) when we apply it to an unsigned graph!

Frackets and Fixed Points

If $\vec{s} \in \mathrm{sstab}(G)$ is a configuration where $LM^{-1}(2\vec{s} - \vec{c}_{\max})$ is integral, then the map μ in the signed duality maps $\vec{s}$ to itself; we call such a configuration a **fixed point**. To analyze the number of fixed points of a signed graph, we will study structures called frackets.

Given a chip-firing pair (L, M) and a fractional vector $\vec{f}$, the **L-fracket** $F_{\vec{f}}^L$ is the subset of $\mathcal{K}(L)$ consisting of every equivalence class that has a vector representation $\vec{v} \in \mathbb{Z}^n$ such that $ML^{-1}\vec{v}$ has fractional part $\vec{f}$.

The **zero fracket** F_0 of a chip firing pair (L, M) is the collection of integral vectors $\vec{v}$ such that $LM^{-1}\vec{v}$ is also integral. Then, the configuration $\vec{s}$ is a fixed point if and only if $\vec{c}_{\max} - 2\vec{s} \in F_0$. The first step to finding the number of fixed points is to study the size of the zero fracket:

Theorem 3. *Let (L, M) be any chip-firing pair. Let p_M be the product of the invariant factors of $\mathcal{K}(M)/F_0^M$ excluding the largest invariant factor, and let p_L be the product of the invariant factors of $\mathcal{K}(M)/F_0^M$ excluding the largest invariant factor. Then, $|F_0^L| = \frac{\gcd(|L|ML^{-1}|, |M|LM^{-1}|)}{\gcd(p_M, p_L)}$.*

The size of the zero fracket is related to the number of fixed points:

Proposition 1. *If there are solutions to $\vec{c}_{max} - 2\vec{s} \in F_0$, then the number of unique solutions up to firing-equivalence is equal to $|F_0|d$, where d is the number of elements of $\mathcal{K}(G)/F_0$ with order at most 2.*

Both of the above results are most useful when we have a guarantee that $\mathcal{K}(G)/F_0$ is cyclic—for example, when $\mathcal{K}(G)$ is cyclic. When this occurs, we can apply ***Theorem 3*** with $p = 1$ to find the size of F_0, and we can also easily find the number of elements of $\mathcal{K}(G)/F_0$ with order at most 2 depending on whether $\mathcal{K}(G)/F_0$ has odd or even order.

References

1. Cho, M. *et al. Chip-firing and critical groups of signed graphs.* 2024.
2. Benton, Z., Kwak, J., Oh, S., Torres, M. & Xie, M. *On z-Superstable and Critical Configurations of Chip Firing Pairs.* https://arxiv.org/abs/2412.02679. 2024.

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