RHOMBIC TABLEAUX FOR PARTIAL FLAG VARIETIES

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BACKGROUND

Schubert polynomials arise in many areas of algebra, geometry and combinatorics. They generalize Schur polynomials which arise from the cohomology of the Grassmannian and are given combinatorially in terms of semi standard Young Tableaux (SSYT).

We give a tableau-like model for Schubert polynomials corresponding to any choice of partial flag variety, which generalizes the SSYT model for Schur polynomials.

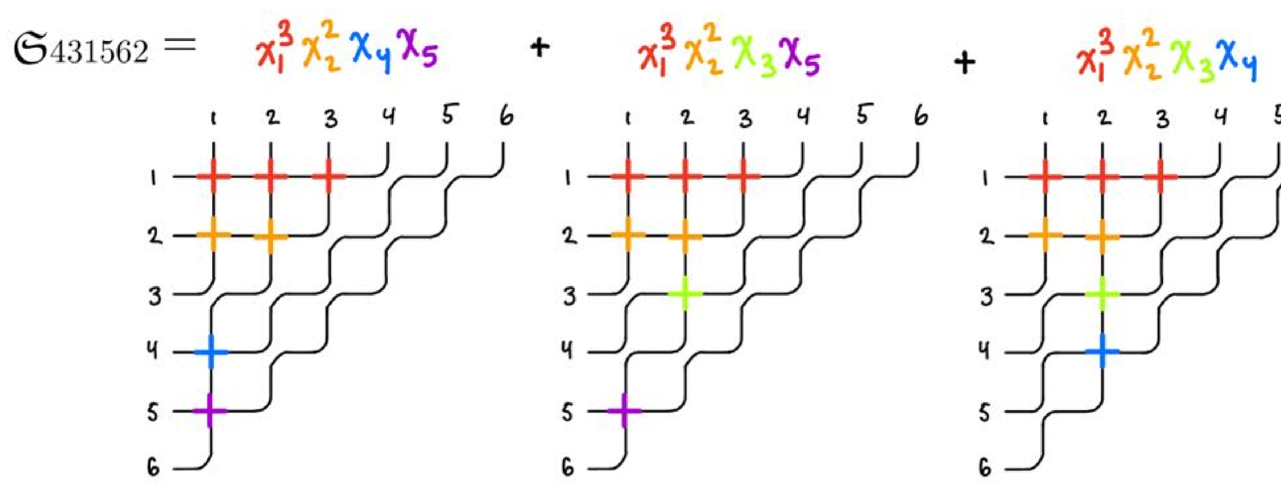
Definition

A pipe dream for $w \in S_n$, is a tiling of all squares in $\{(i,j) | i+j \leq n+1\}$ with either a ⌐ or +, so that the resulting pipe starting at spot i on the left (the i^{th} pipe) ends at spot w_i on the top . It is reduced if no pair of strands cross each other more than once.
Let $\mathcal{RP}(w)$ denote the set of reduced pipe dreams for w.

Theorem (Bergeron and Billey [1])

For a permutation $w \in S_n$, the *Schubert polynomial*

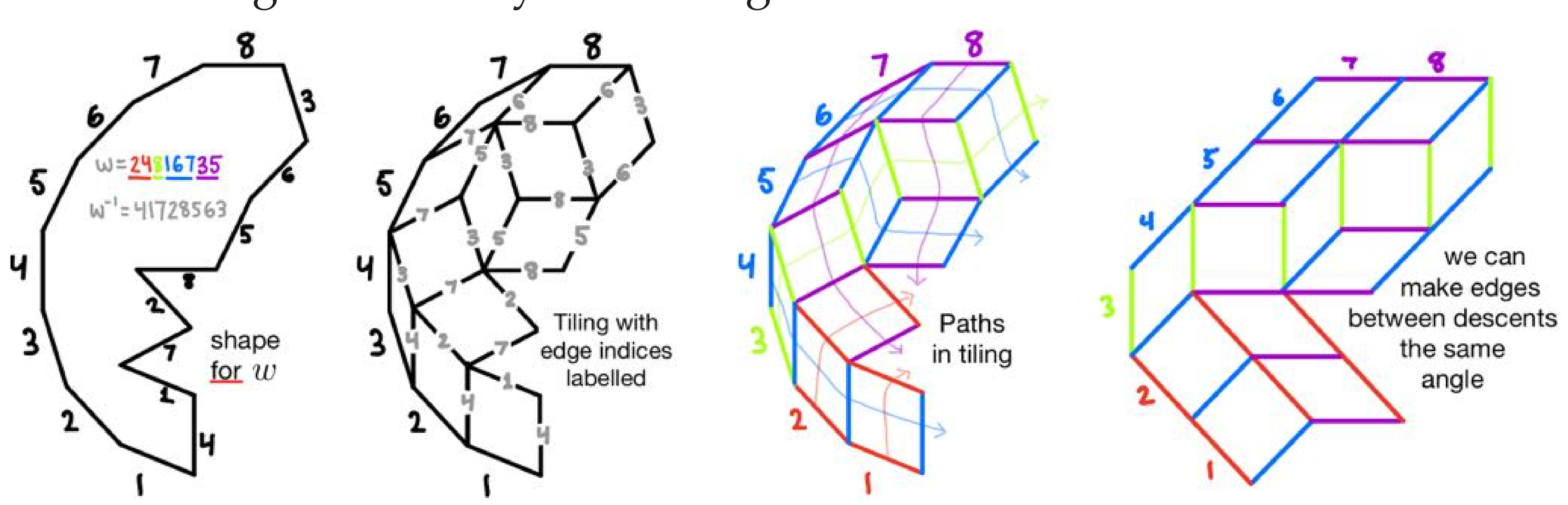
$$\mathfrak{S}_w(x) = \sum_{P \in \mathcal{RP}(w)} \prod_i x_i^{\# \text{ crossings in row } i \text{ of } P}$$



Elnitsky's Tilings (Elnitsky [2])

For $w \in S_n$, draw edges $1, \ldots, n$ in order along the left boundary.
On the right boundary, draw edges in order $w_1^{-1}, \ldots, w_n^{-1}$.
An Elnitsky's tiling for w is a rhombic tiling of this shape.

Assign opposite edges of a rhombus the same index. Let (i,j) be the rhombus with edges indices i and j. The i'th path is the chain of rhombuses from the left to right boundary containing index i.



Spot i is a descent if $w_i > w_{i+1}$. For an index set $I = \{k_1, \cdots, k_l\}$, the parabolic subgroup $W^I = \{w \in S_n :$ All descents occur in spots in $I\}$.
Permutations in W^I correspond to shapes where edges $k_i+1, \ldots, k_{i+1}$ are the same angle for each i.

SEMI STANDARD RHOMBIC TABLEAUX

For an index set $I = (k_1, \ldots, k_l)$ and permutation $w \in W^I$, a Semi Standard Rhombic Tableaux (SSRT) is an Elnitsky tiling of w with numbers $T(i,j) \in \{1, \ldots, n\}$ in each rhombus (i,j) which satisfy the following.

1. Weakly decreasing along paths
2. For $i \leq k_a < j$, we have $T(i,j) \leq k_a$ (Equivalently $T(i,j) \leq i$ for all $i < j$)
3. For $i < j$ and any k, if (i,j) borders (j,k), then $T(i,j) \neq T(j,k)$

Let SSRT(w) be the set of SSRT for w.

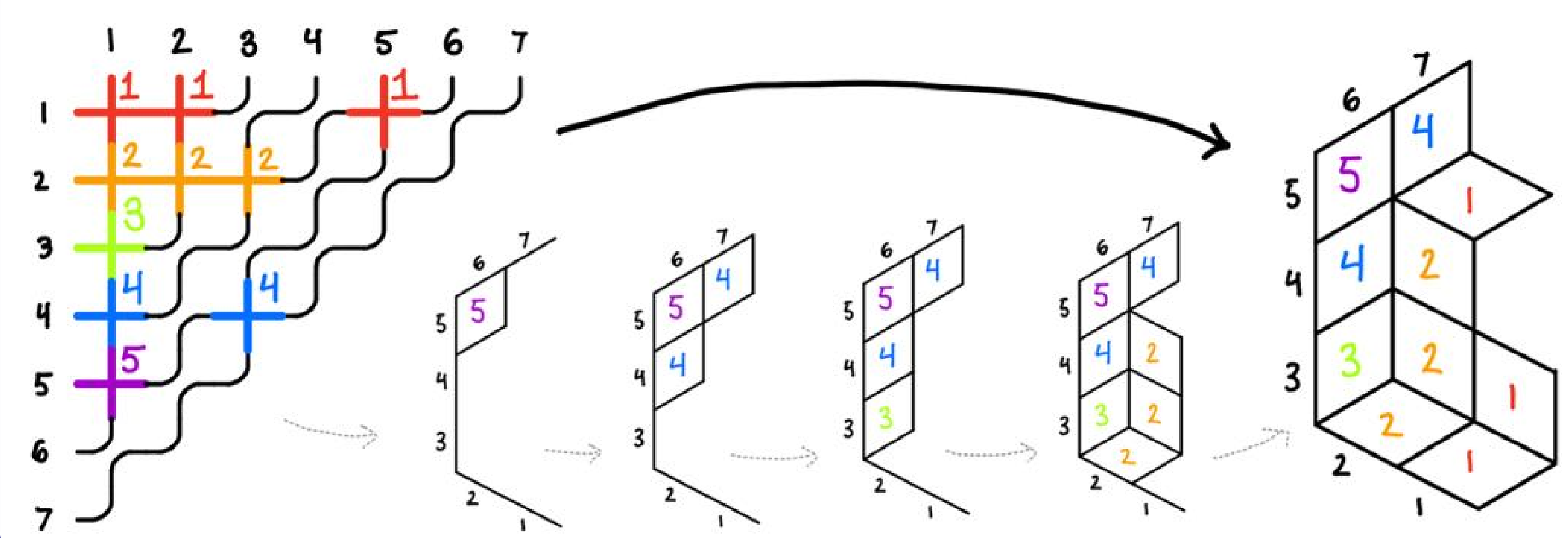
MAIN THEOREM

Theorem

$$\mathfrak{S}_w(x) = \sum_{T \in \text{SSRT}(w)} \prod_i x_i^{\# \text{ of } i\text{'s in } T}$$

Bijection with Pipe Dreams

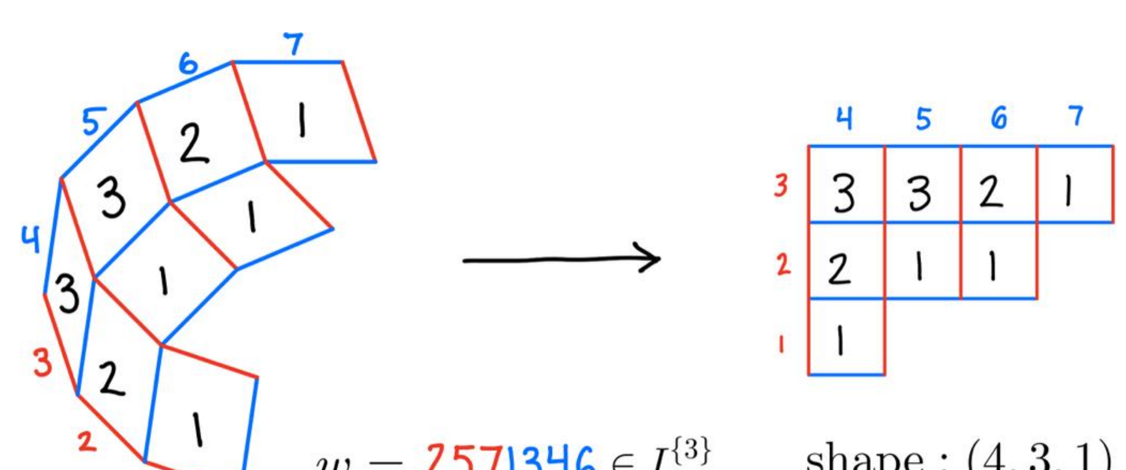
Start with just the left boundary of the SSRT. Read crossings in pipe dream from bottom to top, left to right. When you reach a crossing of pipes i and j in row r, add rhombus (i,j) with entry r. E.x. $w = \underline{36}\,\underline{247}\,\underline{15} \in W^{\{2,5\}}$:



GRASSMANNIAN CASE

For a permutation $w = \underline{a_1 \ldots a_k} b_1 \ldots b_{n-k} \in W^{\{k\}}$ for some k,

$$\text{SSRT}(w) = \left\{ \begin{matrix} \text{reverse SSYT with entries in } \{1, \ldots, \text{k}\} \\ \text{of shape } \lambda_w = (a_k - k, \ldots, a_2 - 2, a_1 - 1) \end{matrix} \right\}$$



GENERALIZATIONS

Theorem: Stanley Symmetric Functions

Let SSRT*(w) be all fillings of Elnitsky tilings for w which satisfy properties 1 and 3 for SSRT. Then the Stanley Symmetric Function for w is

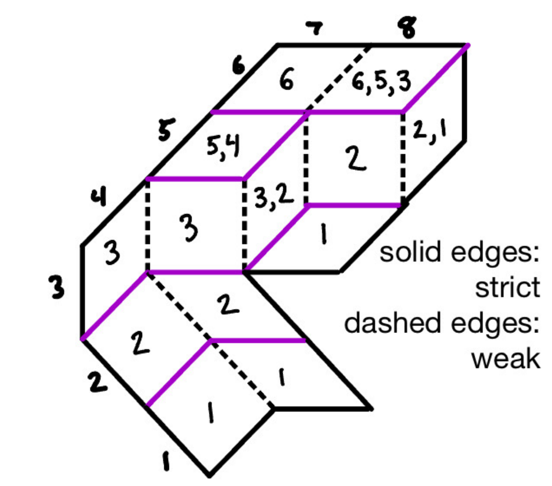
$$F_w = \sum_{T \in \text{SSRT}^*(w)} \prod_i x_i^{\# \text{ of } i\text{'s in } T}$$

Theorem: Set Valued Rhombic Tableaux and k-Theory

Let SVRT(w) be all fillings of Elnitsky's tilings for w with non-empty sets of natural numbers satisfying the same rules as SSRT.

The Grothendeick polynomial for w is

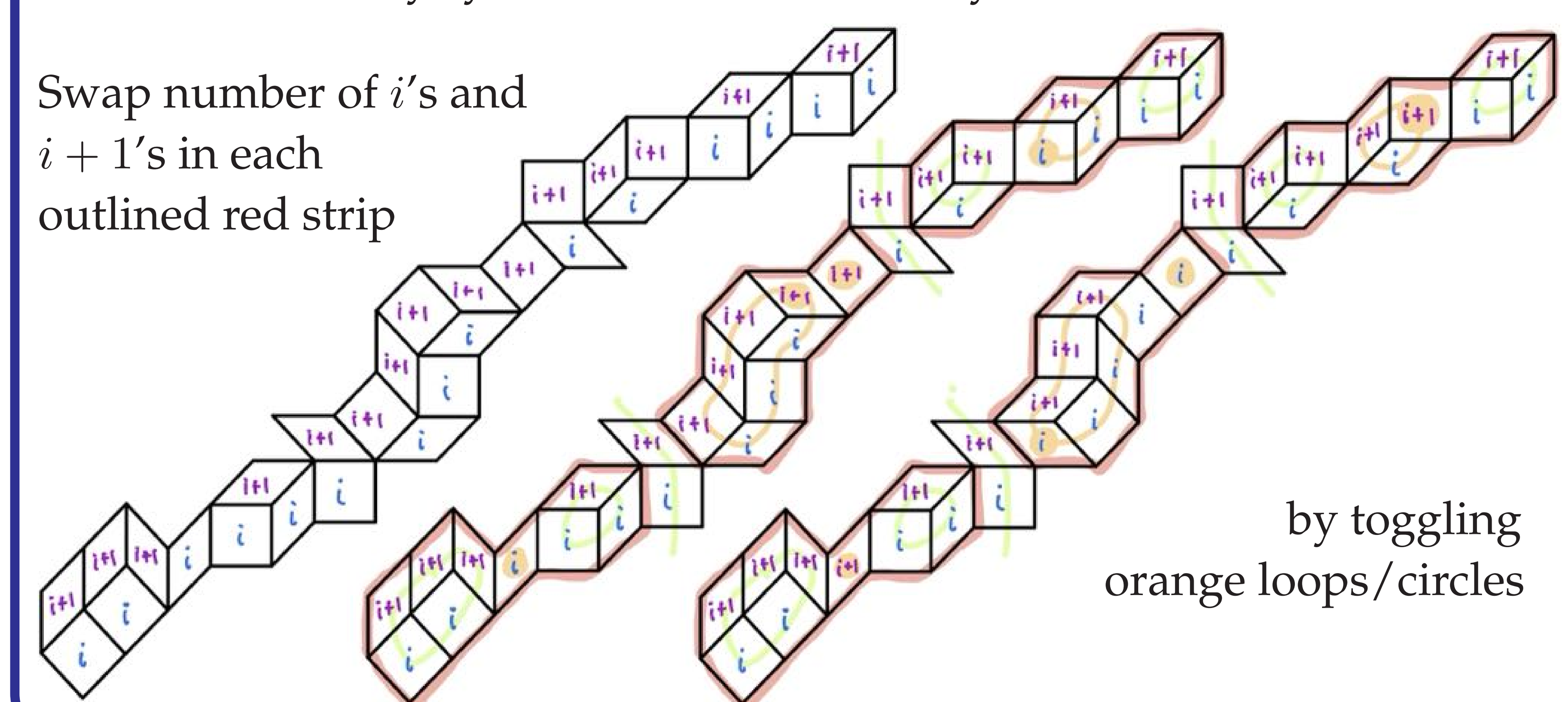
$$\mathfrak{G}_w = \sum_{T \in \text{SSVT}(w)} \prod_i x_i^{\# \text{ of } i\text{'s in } T}$$



BENDER-KNUTH TYPE INVOLUTION ON SSRT

An involution on SSRT*(w) swapping the number of i's and $i+1$'s.
Proves that Stanley symmetric functions are symmetric.

Swap number of i's and $i+1$'s in each outlined red strip



by toggling orange loops/circles

REFERENCES

[1] N. Bergeron and S. Billey. "RC-graphs and Schubert polynomials". *Experimental Mathemat- ics* 2.4 (1993), pp. 257–269.
[2] S. Elnitsky. "Rhombic tilings of polygons and classes of reduced words in Coxeter groups". *Journal of combinatorial theory, Series A* 77.2 (1997), pp. 193–221.

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