Orbit structures and complexity in Schubert and Richardson Varieties

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Abstract

The goal of this work is twofold. Firstly, we provide a type-uniform formula for the torus complexity of the usual torus action on a Richardson variety, by developing the notion of algebraic dimensions of Bruhat intervals. Secondly, when a Levi subgroup in a reductive algebraic group acts on a Schubert variety, we exhibit a codimension preserving bijection between the Levi-Borel subgroup orbits in the big open cell of that Schubert variety and torus orbits in the big open cell of a distinguished Schubert subvariety. This bijection has many applications including a type-uniform formula for the Levi-Borel complexity of a Schubert variety.

Schubert and Richardson Varieties

Let G be a complex, connected, reductive algebraic group with maximal torus T and Borel subgroup B. This data determines the root system Φ, simple roots Δ, and Weyl group W. The homogeneous space G/B is the *full flag variety*.

The B-orbits in G/B are the *Schubert cells*, X_w°, indexed by $w \in W$. Their closures are the *Schubert varieties*, X_w.

Similarly, orbits of the opposite Borel B^- gives *opposite Schubert varieties*, X^w. The intersection of these give *Richardson varieties*, $\mathcal{R}_{u,v} := X_v \cap X^u$.

Complexity of Group Actions

If an algebraic group H acts on a variety X, we say that X is an H-variety. Let H be a reductive algebraic group and B_H a Borel subgroup of H.

Definition: H-Complexity

The *H-complexity* of an H-variety X, denoted $c_H(X)$, is the minimum codimension of a B_H-orbit in X.

Normal H-varieties with $c_H(X) = 0$ are called *H-spherical varieties*. This class generalizes toric varieties. We study the complexity of actions by the torus T and by Levi-Borel subgroups.

Algebraic Dimension of Bruhat Intervals

The *(undirected) Bruhat graph* on W has an edge $w \sim s_\alpha w$ for a positive root $\alpha \in \Phi^+$, with label $\mathrm{wt}(w, s_\alpha w) = \alpha$. For a Bruhat interval $[u,v] := \{w \in W \mid u \leq w \leq v\}$, we define:

Definition ([GH24])

The *algebraic dimension* of $[u,v]$, denoted $\mathrm{ad}(u,v)$, is the dimension of the vector space spanned by all edge labels in the Bruhat graph restricted to $[u,v]$.

This combinatorial statistic governs the geometry of torus orbits. We show that $\mathrm{ad}(u,v)$ can be computed from the root labels of all covers incident to *any* single element $w \in [u,v]$. This provides an efficient computational tool.

Example: Algebraic Dimension

Let $W = S_4$ and consider the interval $[1324, 3412]$. We compute $\mathrm{ad}(1324, 3412)$ using the covers of the maximal element, $v = 3412$. The roots corresponding to the cover relations $w \lessdot v$ are $\{e_1 - e_3, e_2 - e_3, e_1 - e_4, e_2 - e_4\}$. These four vectors span a 3-dimensional space in $\mathbb{R}^4$. Therefore, $\mathrm{ad}(1324, 3412) = 3$.

Torus Complexity

Our first main result is a type-uniform formula for the T-complexity of any Richardson variety.

Theorem 1 ([GH24])

For $u \leq v \in W$, the T-complexity of the Richardson variety is

$$c_T(\mathcal{R}_{u,v}) = \ell(v) - \ell(u) - \mathrm{ad}(u,v).$$

For Schubert varieties ($u = \mathrm{id}$), this simplifies. Let $\mathrm{supp}(w)$ be the number of distinct simple reflections in any reduced word for w.

Corollary ([GH24])

The T-complexity of the Schubert variety X_w is

$$c_T(X_w) = \ell(w) - \mathrm{supp}(w).$$

Levi Subgroup Actions

For $I \subseteq \Delta$, let W_I be the parabolic subgroup of W generated by $\{s_i \mid \alpha_i \in I\}$. The standard parabolic subgroup is $P_I = BW_IB$, with Levi decomposition $P_I = L_I \ltimes U_I$. The group L_I is a *Levi subgroup*, and $B_{L_I} := B \cap L_I$ is a *Levi-Borel subgroup*. An L_I-action on X_w exists if $I \subseteq \mathcal{D}_L(w)$, the *left descent set* of w (i.e., $\{\alpha_i \in \Delta : \ell(s_i w) < \ell(w)\}$).

Orbit Bijection

Let $w = {}_I w\, {}^I w$ be the length-additive left parabolic decomposition of $w \in W$, where ${}_I w \in W_I$. We establish a connection between orbits of B_{L_I} and orbits of T.

Theorem 2 ([GH24])

Let $w \in W$ and $I \subseteq \Delta$. The map

$$\mathfrak{O} : \mathcal{O}_T(X_{{}^I w}^\circ) \to \mathcal{O}_{B_{L_I}}(X_w^\circ)$$

given by $\Theta \mapsto B_{L_I}\, {}^I w x$, where $x \in \Theta$, is a surjection. If L_I acts on X_w, then $\mathfrak{O}$ is a codimension preserving bijection.

This allows us to transfer problems about B_{L_I}-orbits to the more understood setting of T-orbits.

Levi Complexity

Applying the orbit bijection and our torus complexity results, we obtain the following theorem.

Theorem 3 ([GH24])

Let $w \in W$ and suppose L_I acts on X_w. Then the L_I-complexity is given by

$$c_{L_I}(X_w) = \ell({}^I w) - \mathrm{supp}({}^I w).$$

Context and Previous Work

Karuppuchamy provided a succinct classification of toric Schubert varieties [Kar13]. For type A, Lee, Masuda, and Park classified complexity-one Schubert varieties, while Donten-Bury, Escobar, and Portakal computed the torus complexity of Richardson varieties [DBEP23]. Our results provide type-uniform formulas for these complexities.

The study of Levi-actions on Schubert varieties was initiated in [HY22], leading to a classification of L_I-spherical Schubert varieties in [GHY24]. Our Theorem 3 provides a general formula for the L_I-complexity, extending that classification.

Partial Flag Varieties

For $J \subseteq \Delta$, let P_J be the corresponding standard parabolic subgroup. Our results extend to Schubert varieties X_w^J in the *partial flag variety* G/P_J. Here $w \in W^J$, the set of minimal length coset representatives for W/W_J.

Theorem 4 ([GH24])

Let $w \in W^J$. The T-complexity of X_w^J is equal to the T-complexity of the corresponding Schubert variety X_w in the full flag variety G/B:

$$c_T(X_w^J) = c_T(X_w) = \ell(w) - \mathrm{supp}(w).$$

This gives a new, type-uniform classification of toric Schubert varieties in any partial flag variety, generalizing the classification for the full flag variety ($J = \emptyset$) from [Kar13].

Selected References

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