Generalized Laplacian tilings and root zonotopes

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Construction

For a pair A, B of equi-oriented $r \times n$ matrices, there exist tilings of $\mathbb{R}^r$ whose tiles are given by the bases A_σ of A, for $\sigma \subset [n]$, and appear with frequencies equal to the volumes $\mathrm{Vol}(B_\sigma)$.

Note: When all frequencies are integers, the tiling is periodic and its fundamental domain has volume equal to $\det(L)$ for the Laplacian $L := AB^T$.

Notation

- σ is a size r subset of $[r+n]$, i.e., $\sigma \in \binom{[r+n]}{r}$
- $\widehat{\sigma} = [r+n] \setminus \sigma$
- A_σ is the matrix A after restricting to columns in σ
- $\widetilde{B}$ is a matrix whose row space is *dual* to the row space of B
- γ is a vector in $\mathbb{R}^{r+n}$ satisfying certain genericity conditions
- For each $\sigma \in \binom{[r+n]}{r}$ and $\mathbf{z} \in \mathbb{Z}^r$, let

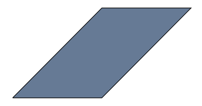
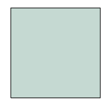
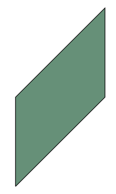
$$\mathbf{s}_{(\sigma,\mathbf{z})} := A_\sigma(\mathbf{z} - \gamma_\sigma) + A_{\widehat{\sigma}}\left(\left\lceil \widetilde{B}_{\widehat{\sigma}}^{-1}\widetilde{B}_\sigma(\gamma_\sigma - \mathbf{z}) + \gamma_{\widehat{\sigma}} \right\rceil - \gamma_{\widehat{\sigma}}\right)$$

How to form a tiling

For each $\sigma \in \binom{[r+n]}{r}$ and $\mathbf{z} \in \mathbb{Z}^r$, draw the parallelogram spanned by the columns of A_σ, translated by $s_{(\sigma,\mathbf{z})}$.

For a certain choice of A and B relating to fifth roots of unity, our construction gives a *generalized Penrose tiling*. Above are two versions of the tiling, which differ only in the generic vector γ.

Constructing tilings from the matrix $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

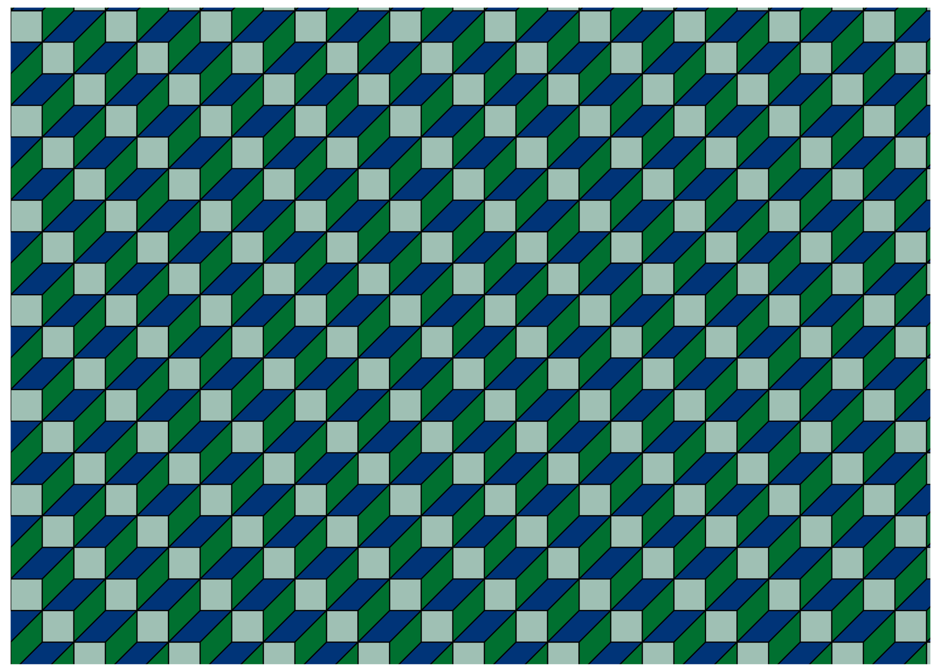
$A_{\{1,2\}} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} =$  $A_{\{1,3\}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} =$  $A_{\{2,3\}} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} =$ 

The matrix A determines the tiles. We also need a matrix B to determine the relative frequencies of each tile. We can choose any matrix B whose maximal minors have the same signs as the maximal minors of A (in this example, all positive).

$B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

$|B_{\{1,2\}}| : |B_{\{1,3\}}| : |B_{\{2,3\}}| =$

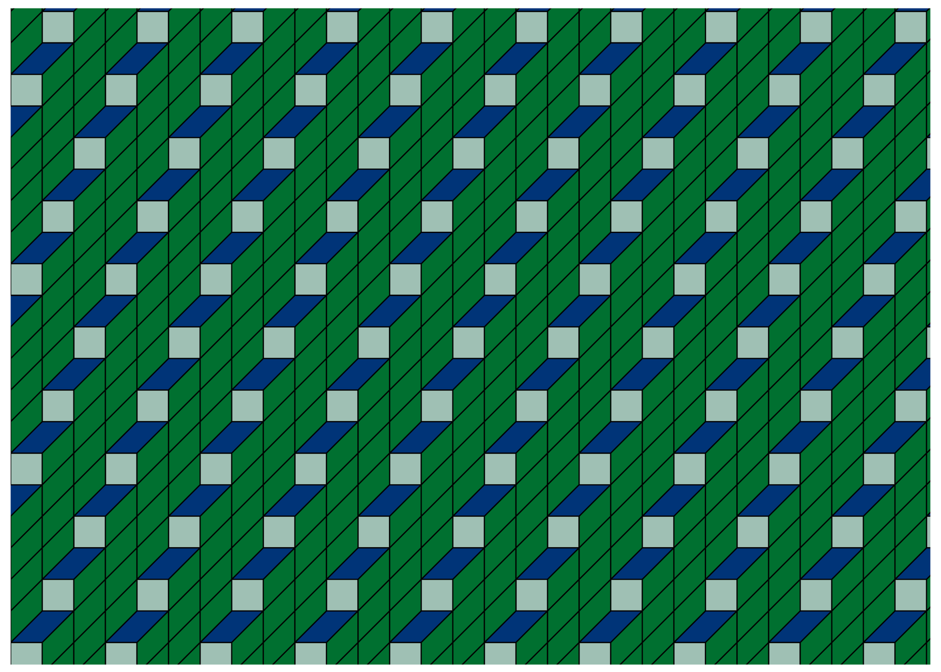
$1 : 1 : 1$



$B = \begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

$|B_{\{1,2\}}| : |B_{\{1,3\}}| : |B_{\{2,3\}}| =$

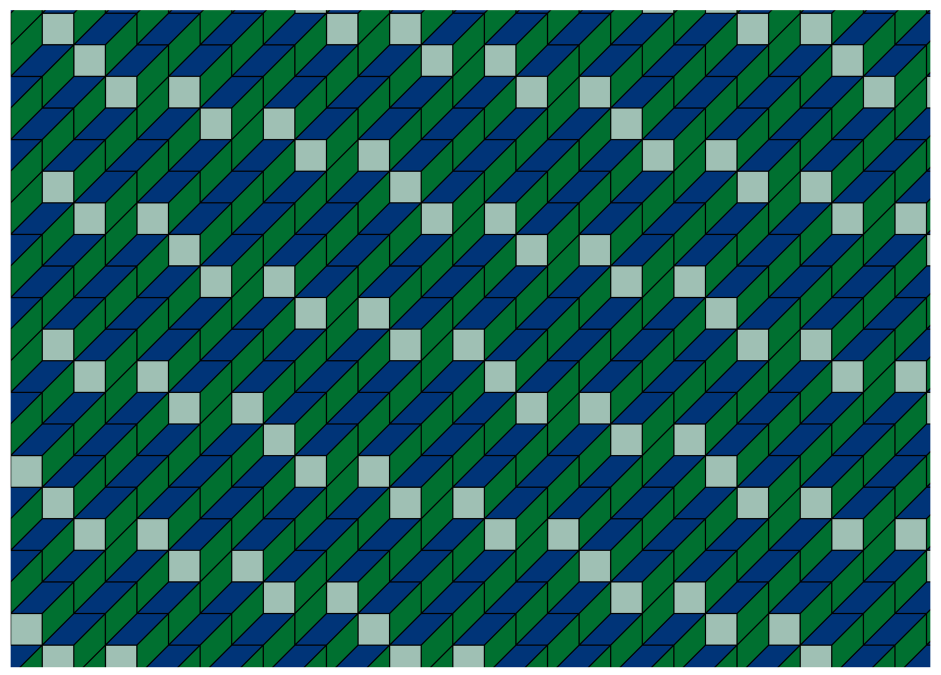
$1 : 1 : 4$



$B = \begin{bmatrix} 1 & \pi & 0 \\ 0 & e & 1 \end{bmatrix}$

$|B_{\{1,2\}}| : |B_{\{1,3\}}| : |B_{\{2,3\}}| =$

$e : 1 : \pi$

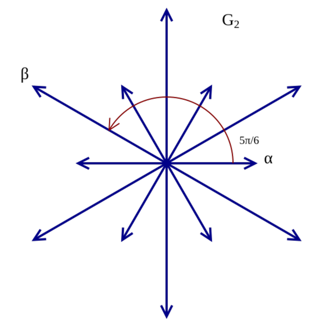


Tilings via root zonotopes and their relatives

When the pair A, B consists of the roots and coroots associated to a Weyl group W, the resulting tiling $\mathcal{T}(W)$ encodes Coxeter combinatorics for W and its reflection subgroups.

- Note: W has rank n, Coxeter number h, connection index $I(W)$.
- The fundamental domain of the tiling $\mathcal{T}(W)$ has volume h^n.
- The probability that a random point belongs to a translate of some tile $\mathcal{T}_\sigma$ equals $\dfrac{I(W_\sigma)}{h^n}$. (W_σ is the σ-induced subgroup of W)

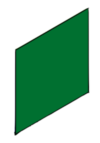
Below is the case $W = G_2 = I_2(6)$, with simple roots

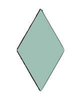


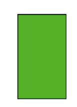
$$\alpha := \sqrt{3} \cdot \begin{bmatrix} \sqrt[4]{4/3} \\ 0 \end{bmatrix} \quad \text{and} \quad \beta := \frac{1}{\sqrt{3}} \cdot \begin{bmatrix} -\sqrt[4]{27/4} \\ \sqrt[4]{3/4} \end{bmatrix}$$

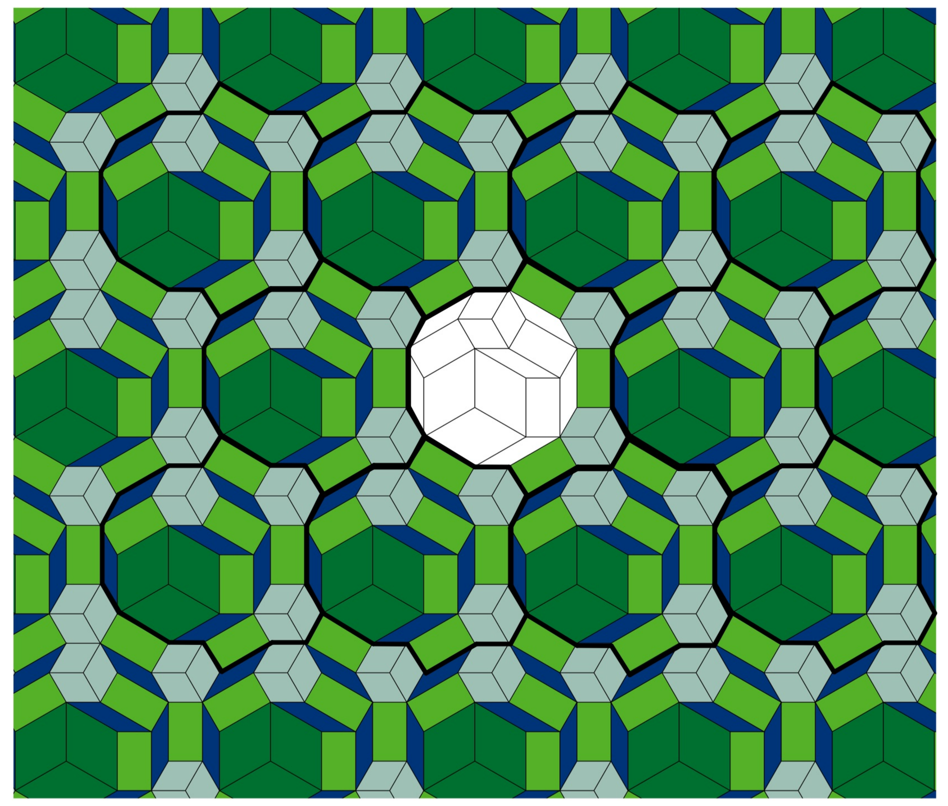
$$A^T := \begin{bmatrix} \alpha \\ \beta \\ \alpha+\beta \\ 2\alpha+\beta \\ 3\alpha+\beta \\ 3\alpha+2\beta \end{bmatrix}$$

$A_{12,23,36,64,45,51}$:

$A_{26,25,56}$: 

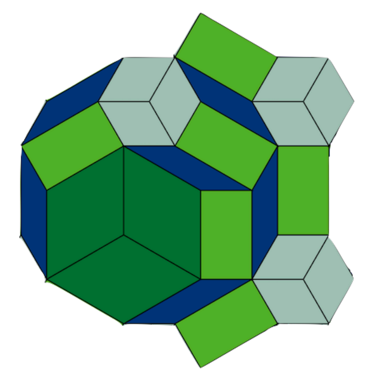
$A_{13,14,34}$: 

$A_{16,24,35}$: 



Note that $(vol) \cdot (freq) = \dfrac{I(W_\sigma)}{h^n}$:

- $A_{12}: \quad 1 \cdot \frac{1}{36} = \frac{I(G_2)}{6^2} \qquad A_{26}: \quad 3 \cdot \frac{1}{36} = \frac{I(A_2)}{6^2}$
- $A_{13}: \quad 1 \cdot \frac{3}{36} = \frac{I(A_2)}{6^2} \qquad A_{16}: \quad 2 \cdot \frac{2}{36} = \frac{I(A_1 \times A_1)}{6^2}$



References

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