Commutative Properties of Schubert Puzzles with Convex Polygonal Boundary Shapes

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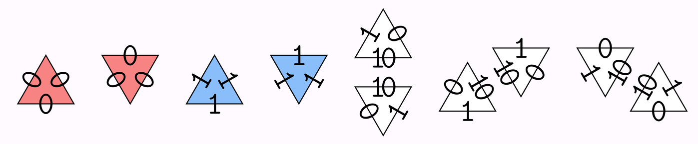
Classical Schubert puzzles

Classically, a **Schubert puzzle** is a tiling of an equilateral trianglar region using a set of allowed **puzzle pieces**



so that only 0 and 1 labels appear along the outer boundary, not 10s.

An equilateral triangular boundary whose NW, NE, and South sides are labeled clockwise (starting at the SW) with binary strings λ, μ, and ν will be denoted $\triangle_{\lambda,\mu,\nu}$ (see Fig. 1), and a puzzle with this boundary labeling will be called a $\triangle_{\lambda,\mu,\nu}$**-puzzle** (see Fig. 2).

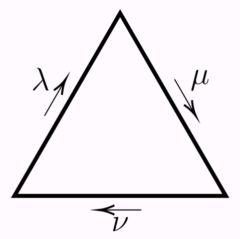


Fig. 1: The labeled boundary $\triangle_{\lambda,\mu,\nu}$

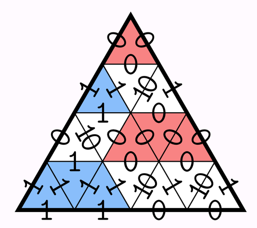


Fig. 2: A $\triangle_{1010,0101,0011}$-puzzle

Puzzles compute Schubert calculus

Classically, **Schubert calculus** is about computing the structure constants in the Schubert variety basis $\{[X_\lambda] : \lambda \in \binom{[n]}{k}\}$ (indexed by binary strings of k 1s and $n-k$ 0s) for the cohomology ring $H^*(\mathrm{Gr}(k;\mathbb{C}^n))$ of the Grassmannian of k-planes in $\mathbb{C}^n$.
These are the coefficients $c^\nu_{\lambda,\mu}$ (Littlewood-Richardson numbers) appearing in the product expansions $[X_\lambda][X_\mu] = \sum_\nu c^\nu_{\lambda,\mu}[X_\nu] \in H^*(\mathrm{Gr}(k;\mathbb{C}^n))$. They can be found *geometrically* as the intersection number of three Schubert varieties:

$$c^\nu_{\lambda,\mu} = \int_{\mathrm{Gr}(k;\mathbb{C}^n)} [X_\lambda][X_\mu][X^\nu].$$

They can also be found *combinatorially* by counting puzzles. Letting $^\vee$ denote reversal, we have:

Theorem ([KTW04], 1999).

$$c^\nu_{\lambda,\mu} = \#\{\triangle_{\lambda,\mu,\nu^\vee}\text{-puzzles}\}.$$

Generalized polygonal Schubert puzzles

Now we generalize the definition of "puzzles" to include puzzle piece tilings of convex polygonal shapes, where

1. the angle of each puzzle piece edge relative to the x-axis is a multiple of 60°, and
2. only 0 and 1 labels are allowed to appear along the outer boundary, not 10s.

This allows us to have puzzles with trapezoidal, parallelogram-shaped, pentagonal, and hexagonal boundary as well. We again use a shape symbol and the suffix -puzzle to denote these puzzles.

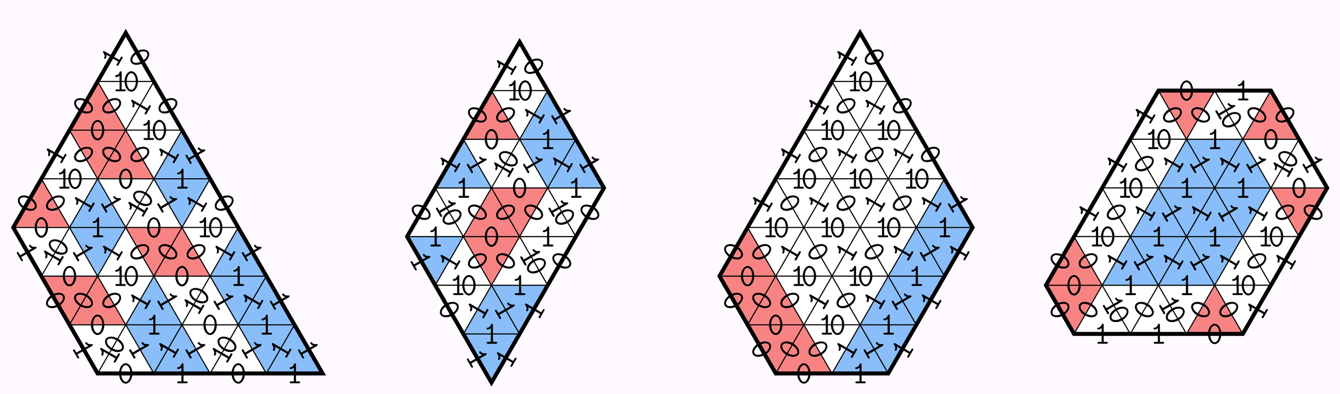
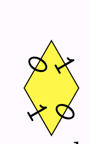


Fig. 3: Examples of a ⏢$_{101,0101,0010111,1010}$-puzzle, a $\lozenge_{101,0101,011,0011}$-puzzle, a ⬠$_{00,01111,0001,111,10}$-puzzle, and a ⬡$_{0,0111,01,01,011,011}$-puzzle.

Equivariant puzzles

An **equivariant puzzle** is one where we additionally allow the *equivariant piece* . This piece carries a special weight of the form $y_j - y_i$, where $1 \le i < j \le n$ and (i,j) corresponds uniquely to the piece's position in the puzzle.

Equivariant puzzles compute Schubert calculus in H^*_T

The classes defined by the Schubert varieties, $\{[X_\lambda] : \lambda \in \binom{[n]}{k}\}$, also form a basis for the T-equivariant cohomology $H^*_T(\mathrm{Gr}(k;\mathbb{C}^n))$.

Theorem ([KT03], 2001). The structure constants in $H^*_T(\mathrm{Gr}(k;\mathbb{C}^n))$ are given by

$$(c_T)^\nu_{\lambda,\mu} = \sum_{\triangle_{\lambda,\mu,\nu^\vee}\text{-puzzles } P} \mathrm{weight}(P) = \sum_{\triangle_{\lambda,\mu,\nu^\vee}\text{-puzzles } P} \left(\prod_{\substack{\text{equivariant}\\ \text{pieces } p \text{ in } P}} \mathrm{weight}(p) \right).$$

Results on commutative properties of polygonal puzzles

The commutative property of classical triangular puzzles says that

$$\#\{\triangle_{\lambda,\mu,\nu}\text{-puzzles}\} = \#\{\triangle_{\mu,\lambda,\nu}\text{-puzzles}\}.$$

(See Fig. 4). We generalize this property to polygonal puzzles in the following theorems.

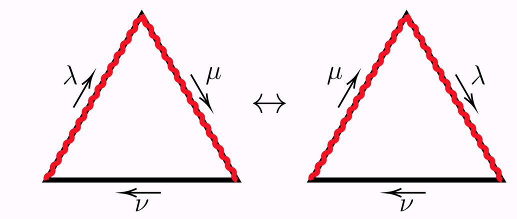
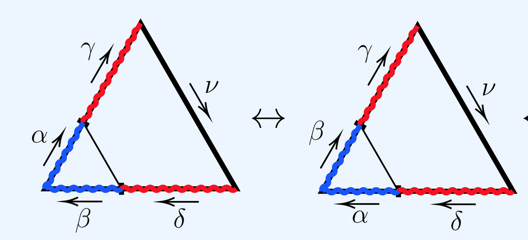


Fig. 4: The NW and NE labels (indicated by red squiggly lines) commute.

Theorems ([And24]). For each labeled boundary shape drawn below, we can commute the labels on any pair of sides with matching colored squiggly lines while preserving the number of puzzles filling the boundary. Namely,

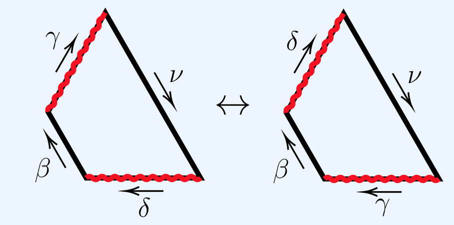
(a) For $\alpha,\beta \in \binom{[a]}{a_1}$, $\gamma,\delta \in \binom{[c]}{c_1}$, and $\nu \in \binom{[c+a]}{a_1+c_1}$,

$$\#\{\triangle_{\alpha\gamma,\nu,\delta\beta}\text{-puzzles}\} = \#\{\triangle_{\beta\gamma,\nu,\delta\alpha}\text{-puzzles}\}$$
$$= \#\{\triangle_{\alpha\delta,\nu,\gamma\beta}\text{-puzzles}\} = \#\{\triangle_{\beta\delta,\nu,\gamma\alpha}\text{-puzzles}\}.$$



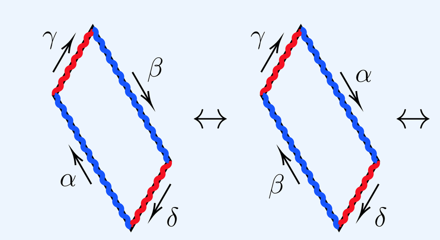
(b) For $\beta \in \binom{[a]}{a_1}$, $\gamma,\delta \in \binom{[c]}{c_1}$, and $\nu \in \binom{[c+a]}{a_1+c_1}$,

$$\#\{⏢_{\beta,\gamma,\nu,\delta}\text{-puzzles}\} = \#\{⏢_{\beta,\delta,\nu,\gamma}\text{-puzzles}\}.$$



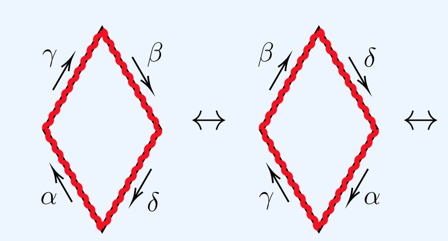
(c) For $\alpha,\beta \in \binom{[a]}{a_1}$ and $\gamma,\delta \in \binom{[c]}{c_1}$,

$$\#\{\lozenge_{\alpha,\gamma,\beta,\delta}\text{-puzzles}\} = \#\{\lozenge_{\beta,\gamma,\alpha,\delta}\text{-puzzles}\}$$
$$= \#\{\lozenge_{\alpha,\delta,\beta,\gamma}\text{-puzzles}\} = \#\{\lozenge_{\beta,\delta,\alpha,\gamma}\text{-puzzles}\}.$$



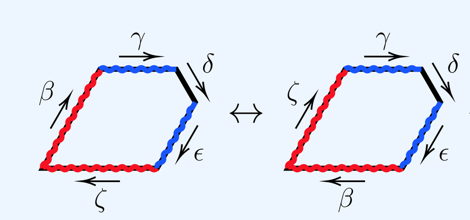
(d) For $\alpha,\beta,\gamma,\delta \in \binom{[a]}{a_1}$ and any bijection $f : \{\alpha,\beta,\gamma,\delta\} \to \{\alpha,\beta,\gamma,\delta\}$,

$$\#\{\lozenge_{\alpha,\gamma,\beta,\delta}\text{-puzzles}\} = \#\{\lozenge_{f(\alpha),f(\gamma),f(\beta),f(\delta)}\text{-puzzles}\}.$$



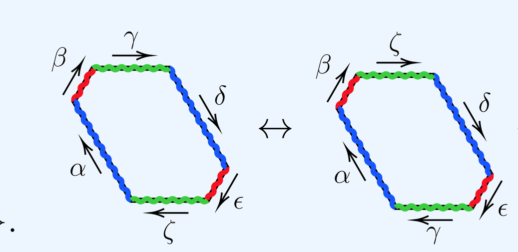
(e) For $\beta,\zeta \in \binom{[b]}{b_1}$, $\gamma,\epsilon \in \binom{[c]}{c_1}$, and $\delta \in \binom{[d]}{d_1}$,

$$\#\{⬠_{\beta,\gamma,\delta,\epsilon,\zeta}\text{-puzzles}\} = \#\{⬠_{\zeta,\gamma,\delta,\epsilon,\beta}\text{-puzzles}\}$$
$$= \#\{⬠_{\beta,\epsilon,\delta,\gamma,\zeta}\text{-puzzles}\} = \#\{⬠_{\zeta,\epsilon,\delta,\gamma,\beta}\text{-puzzles}\}.$$



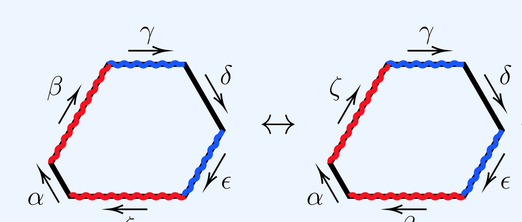
(f) For $\alpha,\delta \in \binom{[a]}{a_1}$, $\beta,\epsilon \in \binom{[b]}{b_1}$, and $\gamma,\zeta \in \binom{[c]}{c_1}$, and any bijections $f : \{\alpha,\delta\} \to \{\alpha,\delta\}$, $g : \{\beta,\epsilon\} \to \{\beta,\epsilon\}$, and $h : \{\gamma,\zeta\} \to \{\gamma,\zeta\}$,

$$\#\{⬡_{\alpha,\beta,\gamma,\delta,\epsilon,\zeta}\text{-puzzles}\} = \#\{⬡_{f(\alpha),g(\beta),h(\gamma),f(\delta),g(\epsilon),h(\zeta)}\text{-puzzles}\}.$$



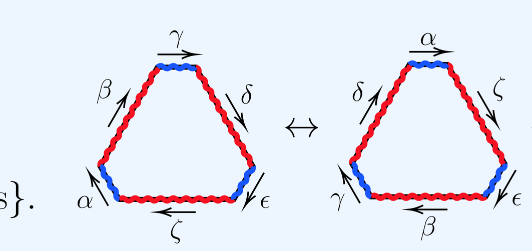
(g) For $\alpha \in \binom{[a]}{a_1}$, $\beta,\zeta \in \binom{[b]}{b_1}$, $\gamma,\epsilon \in \binom{[c]}{c_1}$, and $\delta \in \binom{[d]}{d_1}$,

$$\#\{⬡_{\alpha,\beta,\gamma,\delta,\epsilon,\zeta}\text{-puzzles}\} = \#\{⬡_{\alpha,\zeta,\gamma,\delta,\epsilon,\beta}\text{-puzzles}\}$$
$$= \#\{⬡_{\alpha,\beta,\epsilon,\delta,\gamma,\zeta}\text{-puzzles}\} = \#\{⬡_{\alpha,\zeta,\epsilon,\delta,\gamma,\beta}\text{-puzzles}\}.$$



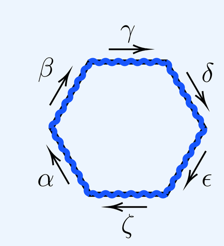
(h) For $\alpha,\gamma,\epsilon \in \binom{[a]}{a_1}$ and $\beta,\delta,\zeta \in \binom{[b]}{b_1}$, and any bijections $f : \{\alpha,\gamma,\epsilon\} \to \{\alpha,\gamma,\epsilon\}$ and $g : \{\beta,\delta,\zeta\} \to \{\beta,\delta,\zeta\}$,

$$\#\{⬡_{\alpha,\beta,\gamma,\delta,\epsilon,\zeta}\text{-puzzles}\} = \#\{⬡_{f(\alpha),g(\beta),f(\gamma),g(\delta),f(\epsilon),g(\zeta)}\text{-puzzles}\}.$$



(i) For $\alpha,\beta,\gamma,\delta,\epsilon,\zeta \in \binom{[a]}{a_1}$ and any bijection $f : \{\alpha,\beta,\gamma,\delta,\epsilon,\zeta\} \to \{\alpha,\beta,\gamma,\delta,\epsilon,\zeta\}$,

$$\#\{⬡_{\alpha,\beta,\gamma,\delta,\epsilon,\zeta}\text{-puzzles}\} = \#\{⬡_{f(\alpha),f(\beta),f(\gamma),f(\delta),f(\epsilon),f(\zeta)}\text{-puzzles}\}.$$



↔ 719 others

Methods

We can give any ordinary or equivariant convex polygonal puzzle a familiar geometric interpretation via an operation to "complete" it to a triangular puzzle (see Fig. 5). Letting **sort** be the operation of moving all 0s ahead of all 1s in a binary string, we obtain bijections

- $\{⏢_{\beta,\gamma,\nu,\delta}\text{-puzzles}\} \leftrightarrow \{\triangle_{\mathrm{sort}(\beta)\gamma,\nu,\delta\beta}\text{-puzzles}\}$,
- $\{\lozenge_{\alpha,\gamma,\beta,\delta}\text{-puzzles}\} \leftrightarrow \{\triangle_{\mathrm{sort}(\alpha)\gamma,\beta\,\mathrm{sort}(\delta),\delta\alpha}\text{-puzzles}\}$,
- $\{⬡_{\alpha,\beta,\gamma,\delta,\epsilon,\zeta}\text{-puzzles}\} \leftrightarrow \{\triangle_{\mathrm{sort}(\alpha)\beta\gamma,\mathrm{sort}(\gamma)\delta\,\mathrm{sort}(\epsilon),\epsilon\zeta\alpha}\text{-puzzles}\}$.

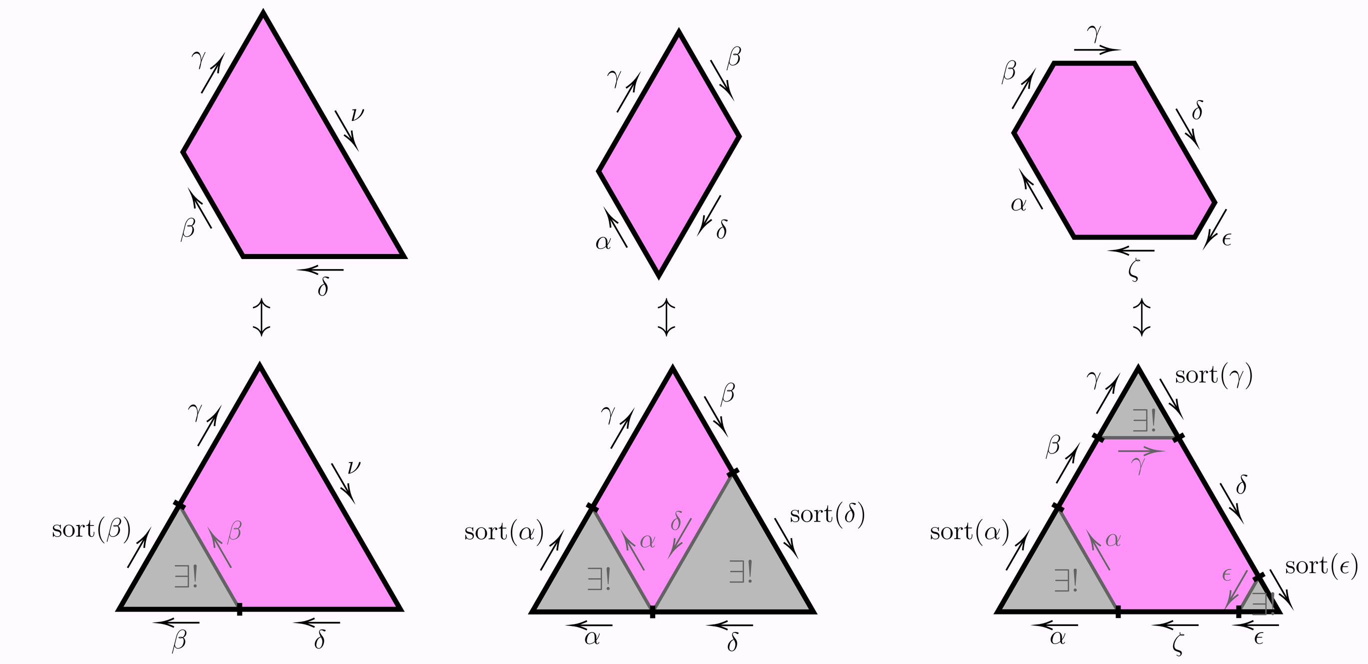


Fig. 5: Given the labels on the triangular boundaries, there exists a unique filling of each grey triangular region with puzzle pieces, which forces the labels around the inner pink regions to replicate those for the polygons on the top row.

Remark. We discovered and proved our results purely within the geometric context provided by this operation, using intersection-theoretic arguments. Though they are primarily combinatorial statements, we currently do not have a direct combinatorial understanding of them.

Result on commutative property of parallelogram-shaped equivariant puzzles

The equivariant structure constants in the theorem below are the ones associated to parallelogram-shaped equivariant puzzles, after completing to a triangle as in Fig. 5.

Theorem ([And24]). Define block matrices

$$\Phi_a := \left[\begin{array}{c|c} J_a & \mathbf{0} \\ \hline \mathbf{0} & I_c \end{array}\right] \quad \text{and} \quad \Phi_c := \left[\begin{array}{c|c} I_a & \mathbf{0} \\ \hline \mathbf{0} & J_c \end{array}\right]$$

in $\mathrm{GL}(\mathbb{C}^{a+c})$, where I_a and J_a (resp. I_c and J_c) denote the $a \times a$ (resp. $c \times c$) identity and anti-diagonal permutation matrices, respectively.
Let $\alpha,\beta \in \binom{[a]}{a_1}$ and $\gamma,\delta \in \binom{[c]}{c_1}$. Then in $H^*_T(\mathrm{Gr}(a_1+c_1;\mathbb{C}^{a+c}))$, we have

$$(c_T)^{(\delta\alpha)^\vee}_{\mathrm{sort}(\alpha)\gamma,\beta\,\mathrm{sort}(\delta)} = \Phi_a\cdot(c_T)^{(\delta\beta)^\vee}_{\mathrm{sort}(\beta)\gamma,\alpha\,\mathrm{sort}(\delta)} = \Phi_c\cdot(c_T)^{(\gamma\alpha)^\vee}_{\mathrm{sort}(\alpha)\delta,\beta\,\mathrm{sort}(\gamma)} = \Phi_c\cdot\Phi_a\cdot(c_T)^{(\gamma\beta)^\vee}_{\mathrm{sort}(\beta)\delta,\alpha\,\mathrm{sort}(\gamma)}.$$

In other words, commuting the pair α,β reverses the $y_1,\dots,y_a$, and commuting the pair γ,δ reverses the $y_{a+1},\dots,y_{a+c}$ in the structure constant.

Corollary. With a further simple proof (not automatically), we also get that the *number* of equivariant puzzles is preserved, i.e.

$$\#\{\lozenge_{\alpha,\gamma,\beta,\delta}\text{-puzzles}\} = \#\{\lozenge_{\beta,\gamma,\alpha,\delta}\text{-puzzles}\} = \#\{\lozenge_{\alpha,\delta,\beta,\gamma}\text{-puzzles}\} = \#\{\lozenge_{\beta,\delta,\alpha,\gamma}\text{-puzzles}\}.$$

Example of commuting the NW and SE labels:

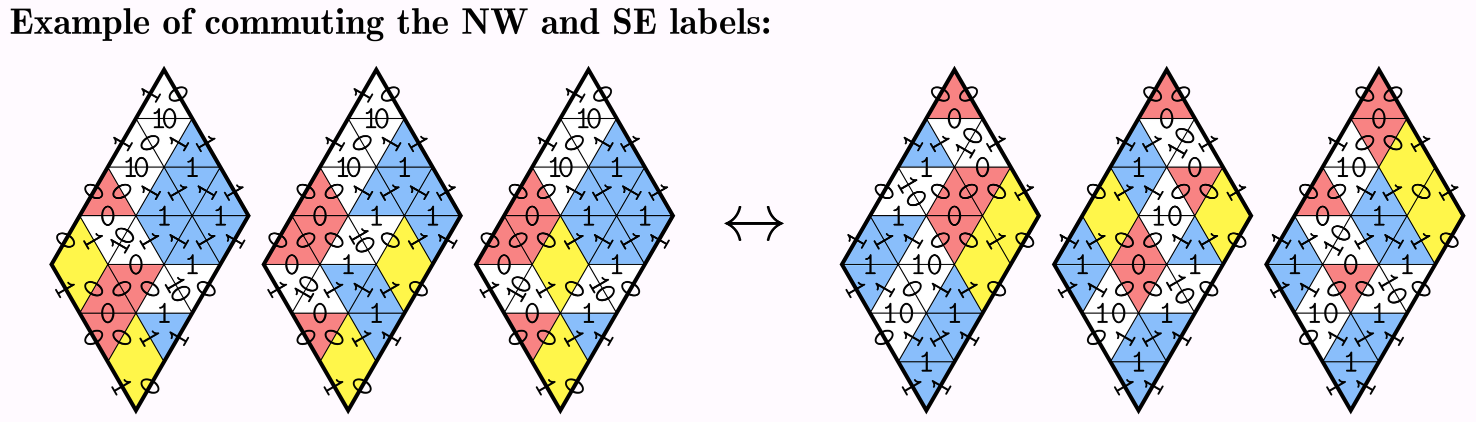


Fig. 6: The sums of the weights on the left and right sides respectively are $(y_4-y_1)(y_4-y_3)+(y_4-y_3)(y_6-y_3)+(y_4-y_3)(y_5-y_2)$ and $(y_7-y_3)(y_6-y_3)+(y_5-y_1)(y_7-y_3)+(y_7-y_3)(y_7-y_2)$. The latter can be obtained by reversing the order of the y_4,y_5,y_6,y_7 in the former.

Remark. We can generalize the theorem beyond the Grassmannian (a 1-step flag manifold) to d-step flag manifolds, i.e. the analogous statement holds in $H^*_T(F\ell(a_1+c_1,a_2+c_2,\dots,a_d+c_d;\mathbb{C}^{a+c}))$.

References

[And24] Portia Anderson. *Commutative Properties of Schubert Puzzles with Convex Polygonal Boundary Shapes*. 2024. arXiv: 2404.06320 [math.CO].

[KT03] Allen Knutson and Terence Tao. "Puzzles and (equivariant) cohomology of Grassmannians". In: *Duke Math. J.* 119.2 (2003), pp. 221–260.

[KTW04] Allen Knutson, Terence Tao, and Christopher Woodward. "The honeycomb model of $\mathrm{GL}_n(\mathbb{C})$ tensor products. II. Puzzles determine facets of the Littlewood-Richardson cone". In: *J. Amer. Math. Soc.* 17.1 (2004), pp. 19–48.

Further questions

1. Extend our results to different types of puzzles, such as those that compute structure constants for K-theory and Segre–Schwartz–MacPherson (SSM) classes, or 2-step, 3-step, and 4-step puzzles, for all types of boundary shape and symmetry.
2. Find a direct combinatorial understanding of our results, perhaps on the level of puzzle pieces.