2025.7 FPSAC Hokkaido University, Sapporo

A combinatorial proof of an identity involving Eulerian numbers

Jerónimo Valencia Porras

UNIVERSITY OF WATERLOO

Abstract

We give a combinatorial proof of an identity involving Eulerian numbers that was obtained algebraically by Brenti and Welker (2009). Our proof is based on alcoved triangulations of dilated hypersimplices. As a byproduct, we describe the dual graph of these triangulations for the dilated standard simplex in terms of words, and conjecture their structure for dilated hypersimplices.

The objects involved

- For $d, r, i \in \mathbb{N}$ and $d \geq 1$, let

$$\mathfrak{C}(r,d,i) := \left\{ \vec{c} \in \mathbb{N}^d \mid c_1 + c_2 + \cdots + c_d = i \ , \ c_j \leq r \text{ for } 1 \leq j \leq d \right\},$$

and denote by $C(r,d,i)$ its cardinality.

- Let $d \geq 1$ and $1 \leq j \leq d$. Define

$$\mathfrak{S}(d,j) := \left\{ \sigma \in \mathfrak{S}_d \mid \operatorname{des}(\sigma) = j-1 \right\},$$

and denote by $A(d,j)$ its cardinality. This is the j-th **Eulerian number**.

The identity involving Eulerian numbers

Proposition 1.1 (Brenti–Welker, 2009):

Let $d, r \geq 1$. Then, for $i = 1, \ldots, d$,

$$r^d A(d,i) = \sum_{j=0}^{d} C(r-1, d+1, ir-j) A(d,j). \tag{1}$$

In particular, when $i = 1$,

$$r^d = \sum_{j=0}^{d} C(r-1, d+1, r-j) A(d,j). \tag{2}$$

"Clearly, [this] proposition asks for a ***combinatorial proof****."* [BW09]

Reinterpreting the identities geometrically

Theorem 1.2 (Attributed to Laplace):

The i-th hypersimplex of dimension d is the polytope given by

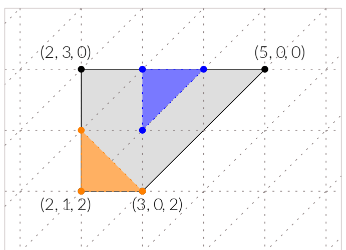
$$\Delta_{i,d} = \left\{ \vec{x} \in \mathbb{R}_x^{d+1} \ : \ 0 \leq x_1, \ldots, x_{d+1} \leq 1 \quad \text{and} \quad x_1 + \cdots + x_{d+1} = i \right\}.$$

The volume of $\Delta_{i,d}$ is given by the i-th Eulerian number, that is $\operatorname{vol}(\Delta_{i,d}) = A(d,i)$.

Therefore $r^d A(d,i) = \operatorname{vol}(r\Delta_{i,d})$, so we want to **understand an unimodular triangulation** of $r\Delta_{i,d}$ in two different ways in order to prove the identities. Luckily, the hypersimplices are **alcoved polytopes!**

Recall that alcoved polytopes come equipped with a unimodular triangulation induced by the affine Coxeter arrangement. Denote by $\mathcal{A}(P)$ the **set of simplices of the alcoved polytope** P **in the alcoved triangulation.** There is a combinatorial description of this set in type A using **sorted sets** [LP07].

Example (Alcoves and sorted sets):



$A = \{(3,2,0), (4,1,0), (3,1,1)\}$
$I_1 = \{1,1,1,1,2\}$
$I_2 = \{1,1,1,2,2\}$
$I_3 = \{1,1,1,2,3\}$

$B = \{(3,0,2), (2,1,2), (2,2,1)\}$
$I_1 = \{1,1,1,3,3\}$
$I_2 = \{1,1,2,3,3\}$
$I_3 = \{1,1,2,2,3\}$

Main Theorem (V.P., 2024)

Let $\mathcal{A}(r\Delta_{i,d})$ be the set of alcoves of the r-dilated hypersimplex $\Delta_{i,d}$. There exist bijections

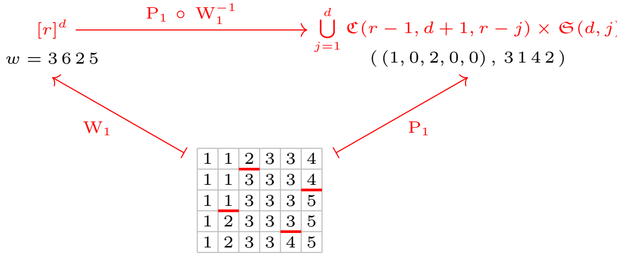
$$\mathrm{W}_i : \mathcal{A}(r\Delta_{i,d}) \longrightarrow [r]^d \times \mathfrak{S}(d,i)$$

$$\mathrm{P}_i : \mathcal{A}(r\Delta_{i,d}) \longrightarrow \bigcup_{j=1}^{d} \mathfrak{C}(r-1, d+1, ir-j) \times \mathfrak{S}(d,j)$$

from which we obtain a combinatorial proof of Equation (1).

The dilated standard simplex $r\Delta_{1,d}$

To prove the Main Theorem, it is useful to start by considering the case of $i = 1$, which we summarize in the following diagram.



Theorem 2.1 (V.P., 2024): The maps

$$\mathrm{W}_1 : \mathcal{A}(r\Delta_{1,d}) \longrightarrow [r]^d$$

$$\mathrm{P}_1 : \mathcal{A}(r\Delta_{1,d}) \longrightarrow \bigcup_{j=1}^{d} \mathfrak{C}(r-1, d+1, r-j) \times \mathfrak{S}(d,j)$$

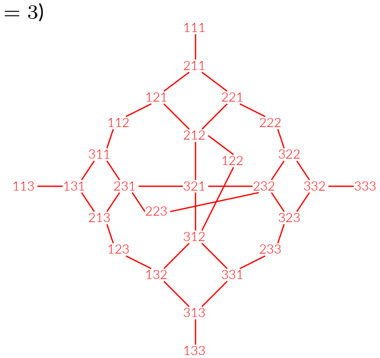
are bijections and give a combinatorial proof of Equation (2).

We can use the map W_1 to understand the alcoved triangulation of $r\Delta_{1,d}$ even further.

Proposition 2.2 (V.P., 2024): The dual graph of the alcoved triangulation of $r\Delta_{1,d}$ is isomorphic to $G_{r,d}$, the graph on vertex set $[r]^d$ and edges given by

- $w_1\, w_2 \ldots w_d \ \sim \ (w_d + 1)\, w_1\, w_2 \ldots w_{d-1}$ whenever $1 \leq w_d < r$, and
- $w_1 \ldots w_i\, w_{i+1} \ldots w_d \ \sim \ w_1 \ldots w_{i+1}\, w_i \ldots w_d$ for any $1 \leq i \leq d-1$ such that $w_i \neq w_{i+1}$.

Example ($r = d = 3$)

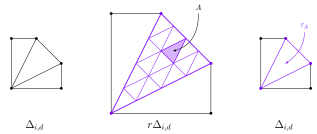


The dilated hypersimplex $r\Delta_{i,d}$

Now we can describe the maps from the **Main Theorem.** First,

$$\mathrm{W}_i : \mathcal{A}(r\Delta_{i,d}) \longrightarrow [r]^d \times \mathfrak{S}(d,i)$$

is computed as follows for $A \in \mathcal{A}(r\Delta_{i,d})$: Find the permutation $\tau_A \in \mathfrak{S}(d,i)$ that labels the alcove of $\Delta_{i,d}$ whose dilation contains A, and then apply W_1 relative to that dilated standard simplex.



On the other hand, the map

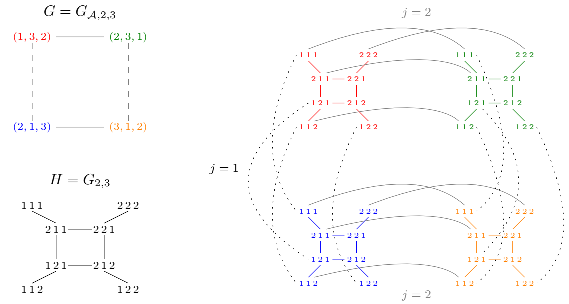
$$\mathrm{P}_i : \mathcal{A}(r\Delta_{i,d}) \longrightarrow \bigcup_{j=1}^{d} \mathfrak{C}(r-1, d+1, ir-j) \times \mathfrak{S}(d,j)$$

is computed directly from the decorated matrix of the alcove.

The structure of the dual graph of the alcoved triangulation of $r\Delta_{i,d}$ is more complicated, but we believe we can recover it as a **composition of dual graphs** (see [VP24, Section 3.2.3]).

Conjecture 3.1 (V.P., 2024): Let $G = G_{\mathcal{A},i,d}$ and $H = G_{r,d}$. The edge-coloring of G determined by the hyperplane types prescribes a choice of connecting sets that make $G\langle H \rangle$ isomorphic to the dual graph of the alcoved triangulation of $r\Delta_{i,d}$.

Example ($i = 2$, $d = 3$ and $r = 2$)



Further and related work

- $\mathrm{P}_i \circ \mathrm{W}_i^{-1} : [r]^d \times \mathfrak{S}(d,i) \longrightarrow \bigcup_{j=1}^{d} \mathfrak{C}(r-1, d+1, ir-j) \times \mathfrak{S}(d,j)$ can be upgraded to a **weight-preserving bijection** to obtain identities for the **q-Eulerian numbers**.
- Based on the work of Lam and Postnikov [LP12] on **alcoved polytopes for crystallographic roots systems**, we can construct a similar combinatorial model for the alcoved triangulation of the dilated C-hypersimplices.
- Ferroni and McGinnis [FM24, Theorem 1.2] give the (positive) coefficients of the **Ehrhart polynomial of slices of prisms** as a sum of products of Eulerian numbers and compatible weighted permutations. **How does this relate to the combinatorial proofs presented?**

References

[BW09] Francesco Brenti and Volkmar Welker. The Veronese construction for formal power series and graded algebras. *Advances in Applied Mathematics*, 42(4):545–556, 2009.

[FM24] Luis Ferroni and Daniel McGinnis. Lattice points in slices of prisms. *Canadian Journal of Mathematics*, pages 1–28, 2024.

[LP07] Thomas Lam and Alexander Postnikov. Alcoved polytopes, I. *Discrete & Computational Geometry*, 38:453–478, 2007.

[LP12] Thomas Lam and Alexander Postnikov. Alcoved Polytopes II. *arXiv:1202.4015*, 2012.

[VP24] Jerónimo Valencia-Porras. A combinatorial proof of an identity involving Eulerian numbers. *arXiv:2410.01179*, 2024.