The Integer Decomposition Property in Smooth Lattice Polytopes

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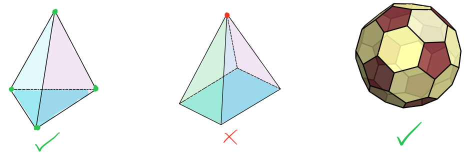
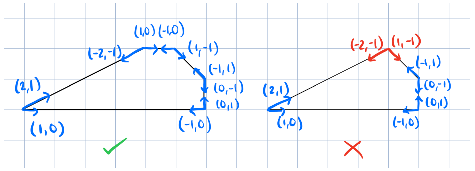
Background

Definition. A d-dimensional polytope is **simple** if each vertex is contained in exactly d edges (and so also exactly d facets).

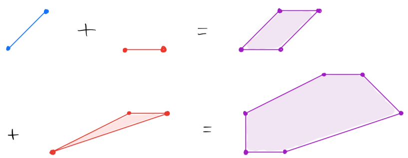


Definition. A d-dimensional polytope is **smooth** if it is simple and the set of primitive edge directions at every vertex spans $\mathbb{Z}^d$.



Definition. The **Minkowski sum** of polytopes P and Q is

$$P + Q = \{p + q : p \in P, q \in Q\}.$$

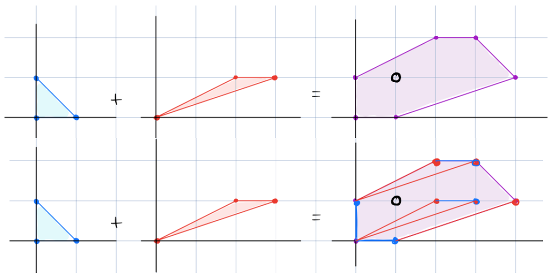


Definition. A pair of d-dimensional polytopes (P, Q) has the **Integer Decomposition Property**, or **is IDP**, when every lattice point in $P + Q$ can be written as the sum of a lattice point in P and a lattice point in Q, i.e.

$$(P + Q) \cap \mathbb{Z}^d = (P \cap \mathbb{Z}^d) + (Q \cap \mathbb{Z}^d).$$

A single polytope P **is IDP** when (P, kP) is IDP for all positive integers k.

Example. The following pair of polytopes is *not* IDP, as one lattice point in their sum cannot be expressed as the sum of a lattice point from each.



Oda's Conjecture

All smooth polytopes are IDP.

Tadao Oda posed this in 1997, it was documented in 2008, and it remains open, even in three dimensions. Recent progress was made towards proving that particular classes of smooth polytopes are IDP.

Motivation. This problem is primarily motivated by the field of commutative algebra as there is a correspondence between smooth polytopes and ample divisors of smooth toric varieties. In addition, IDP polytopes are related to Ehrhart theory and are of use in integer programming.

Recent progress

- In [1] it was shown that 3-dimensional, centrally symmetric, smooth polytopes are IDP.
- In [2], it was shown that smooth 3-dimensional prismatoids have unimodular covers, which implies that they are IDP.

Definition. A **prismatoid** is usually defined to be a polytope whose vertices all lie in two parallel facets. We further require the face poset of a prismatoid be isomorphic to that of a prism.

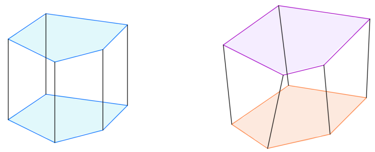


Figure 1. (a) Prism, (b) Prismatoid with isomorphic face poset.

Our result

All smooth combinatorial cubes are IDP.

Definition. A **combinatorial cube** is a polytope whose face poset is isomorphic to that of a cube.

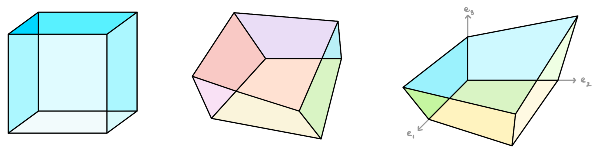
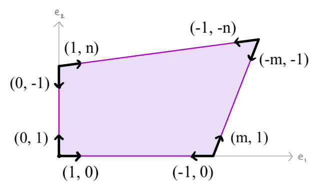


Figure 2. (a) Unit cube, (b) Arbitrary combinatorial cube, (c) Via unimodular transformation, we may assume that one vertex has the standard coordinate vectors as its primitive edge directions.

Main observation

All smooth combinatorial cubes have two parallel facets. Thus, they are prismatoids.

In 2 dimensions:



$$\det \begin{pmatrix} -1 & -m \\ -n & -1 \end{pmatrix} = \pm 1$$

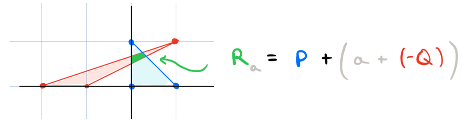
$\Rightarrow n$ or m is 0.

Other pieces of proof

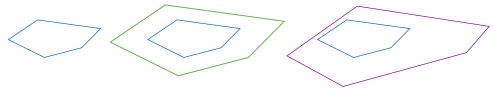
IDP Equivalence. Let P and Q be polytopes. For a lattice point $a \in \mathbb{Z}^d$, define

$$R_a = P \cap (a + (-Q)).$$

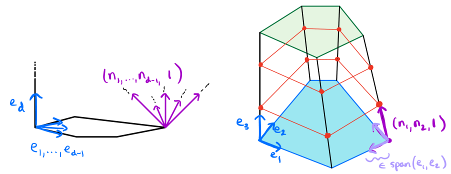
Then, (P, Q) is IDP if and only if for all a, R_a contains a lattice point when it is non-empty.



Definition. Two polytopes are **Minkowski equivalent** when their face posets are isomorphic and corresponding faces are parallel to each other.



Lemma. (Similar result in [2]) All slices of a *smooth* prismatoid are lattice and Minkowski equivalent to each other.



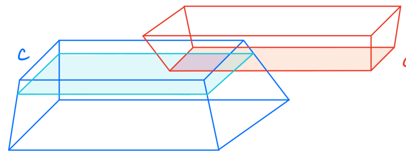
Theorem. ([3]) Let P and P' be Minkowski equivalent lattice polygons. Then, (P, P') is IDP.

Lemma. Let P and P' be smooth, 3-dimensional, Minkowski equivalent lattice prismatoids. Then, (P, P') is IDP.

Final piece

As combinatorial cubes are highly structured, we can use induction to show the following.

Theorem. Suppose that C and C' are Minkowski equivalent smooth combinatorial cubes of dimension $d \geq 2$. Then, (C, C') is IDP.



References

[1] Matthias Beck et al. "Smooth centrally symmetric polytopes in dimension 3 are IDP". In: *Ann. Comb.* 23.2 (2019), pp. 255–262. ISSN: 0218-0006,0219-3094. DOI: 10.1007/s00026-019-00418-x. URL: https://doi.org/10.1007/s00026-019-00418-x.

[2] Giulia Codenotti and Francisco Santos. "Unimodular covers of 3-dimensional parallelepipeds and Cayley sums". In: *Combinatorial Theory* 3.3 (Dec. 2023). ISSN: 2766-1334. DOI: 10.5070/c63362785. URL: http://dx.doi.org/10.5070/C63362785.

[3] Christian Haase et al. "Lattice points in Minkowski sums". In: *arXiv preprint arXiv:0711.4393* (2007).