Quantum Bumpless Pipe Dreams

Tuong Le, Shuge Ouyang, Joseph Restivo, Leo Tao and Angelina Zhang

Department of Mathematics, University of Michigan

Schubert Calculus

Geometric problem
Count the number of points of intersection of some subvarieties of some varieties X.

Algebra
Cohomology of X.

Polynomial representative
Family of polynomials that represents the cohomology of X.

Cohomology	Polynomials
$H_T^*(\mathrm{Fl}_n)$	Double Schubert polynomials
$QH_T^*(\mathrm{Fl}_n)$	Quantum double Schubert polynomials

(Quantum) double Schubert Polynomials

Double Schubert Polynomials

- Represents the torus-equivariant cohomology of the complete flag variety Fl_n.
- Is indexed by permutations in S_n (e.g. $\mathfrak{S}_\sigma(x,y)$ for $\sigma \in S_n$) and lives in $\mathbb{Z}[x,y] = \mathbb{Z}[x_1,\ldots,x_n,y_1,\ldots,y_n]$.
- Has combinatorial formulas for the monomial expansion—pipe dreams (Fomin–Kirillov '96) and bumpless pipe dreams (Lam–Lee–Shimonozo '21).

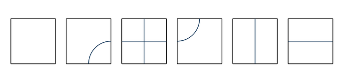
Quantum Schubert Polynomials

- Represents the torus-equivariant quantum cohomology of the complete flag variety Fl_n.
- Is indexed by permutations in S_n (e.g. $\mathfrak{S}^q_\sigma(x,y)$ for $\sigma \in S_n$) and lives in $\mathbb{Z}[x,y,q] = \mathbb{Z}[x,y][q_1,\ldots q_{n-1}]$.
- We discovered a generalization of the bumpless pipe dreams called *quantum bumpless pipe dreams* that give a combinatorial formula for the monomial expansion of quantum double Schubert polynomials.

σ	$\mathfrak{S}_\sigma(x,y)$	$\mathfrak{S}^q_\sigma(x,y)$
123	1	1
213	x_1-y_1	x_1-y_1
132	$x_1+x_2-y_1-y_2$	$x_1+x_2-y_1-y_2$
231	$(x_1-y_1)(x_2-y_1)$	$(x_1-y_1)(x_2-y_1)+q_1$
312	$(x_1-y_1)(x_1-y_2)$	$(x_1-y_1)(x_1-y_2)-q_1$
321	$(x_1-y_1)(x_2-y_1)(x_1-y_2)$	$((x_1-y_1)(x_2-y_1)+q_1)(x_1-y_2)$

Bumpless Pipe Dreams

Definition (Lam–Lee–Shimozono '21). *A bumpless pipe dream (BPD) is a tiling of $n\times n$ grid filled with tiles*



so that

- *The tiling forms n pipes;*
- *Each pipe starts horizontally at the right edge of the grid and ends vertically at the bottom edge of the grid;*
- *No two pipes cross more than once (reduced.)*

Each bumpless pipe dream has an associated permutation σ which is the permutation that maps the row that each pipe starts with to the column that each pipe ends with.

Definition. *The binomial weight of a bumpless pipe dream P, denoted $\mathrm{bwt}(P)$ is*

$$\mathrm{bwt}(P) := \prod_{P(i,j)=\square} (x_i - y_j)$$

Theorem 1: (Lam–Lee–Shimozono '21)

A double Schubert polynomial $\mathfrak{S}_\sigma(x,y)$ is the sum of all binomial weights of all bumpless pipe dreams associated with σ.

Example 1. *From Figure 1,*

$$\mathfrak{S}_{3142}(x,y) = (x_1-y_1)(x_1-y_2)(x_3-y_2) + (x_1-y_1)(x_1-y_2)(x_2-y_1).$$

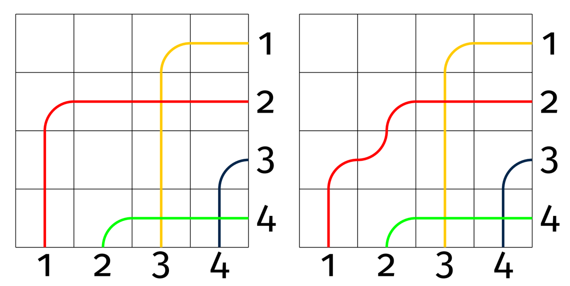


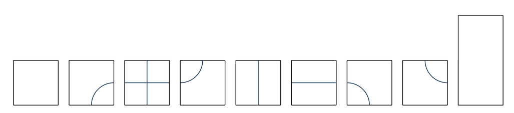
Figure 1: *Bumpless pipe dreams for 3142*

Quantum Bumpless Pipe Dreams

Quantum Bumpless Pipe Dreams

We came up with a construction called *quantum bumpless pipe dreams* which generalize bumpless pipe dreams and give rise to quantum Schubert polynomial.

Definition (Le–Ouyang–Restivo–Tao–Zhang '25). *A quantum bumpless pipe dream (QBPD) is a tiling of an $n\times n$ grid filled with tiles*



so that

- *The tiling forms n pipes;*
- *Each pipe starts horizontally at the right edge of the grid and ends vertically at the bottom edge of the grid;*
- *The pipes only move upward, downward, or leftward (but not rightward) when moving from the right edge to the bottom edge;*
- *No two pipes cross more than once (reduced.)*

Example 2. *Figure 2 is NOT a QBPD since the pipe is moving rightward in the indicated tile. See Example 3 for examples of valid QBPDs.*

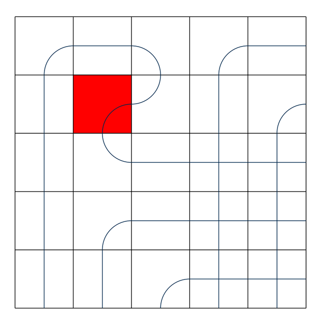


Figure 2: *A non-example of a QBPD*

The binomial weight for a quantum bumpless pipe dream is the product of the following:

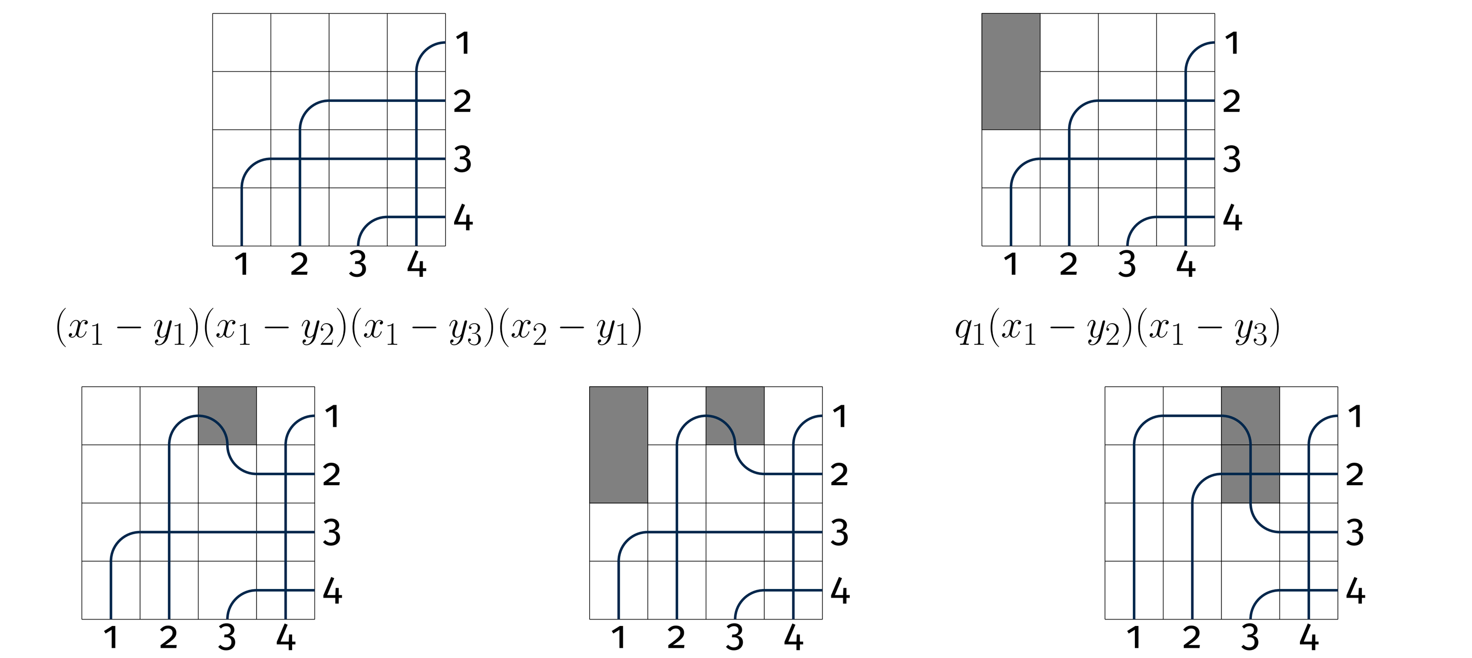
- An empty tile on row i column j contributes $x_i - y_j$
- A domino tile starting on row i contributes q_i
- A cross tile on row i where the vertical strand moves upwards contributes q_i
- A southwest elbow on row i contributes $-q_i$
- A vertical tile on row i where the strand is moving upward contributes $-q_i$

Theorem 2: (Le–Ouyang–Restivo–Tao–Zhang '25)

The quantum double Schubert polynomial for a permutation σ is equal to the sum of the weight of all quantum bumpless pipe dream for σ.

Example 3. *Below are all the QBPDs of 4213 with their binomial weights. Thus we have*

$$\mathfrak{S}^q_{4213}(x,y) = (x_1-y_1)(x_1-y_2)(x_1-y_3)(x_2-y_1) + q_1(x_1-y_2)(x_1-y_3) + (x_1-y_1)(x_2-y_1)(-q_1) + q_1(-q_1) + (-q_1)q_2\,.$$



$(x_1-y_1)(x_1-y_2)(x_1-y_3)(x_2-y_1)$ $\quad$ $q_1(x_1-y_2)(x_1-y_3)$

$(x_1-y_1)(x_2-y_1)(-q_1)$ $\quad$ $q_1(-q_1)$ $\quad$ $(-q_1)q_2$

We give a bijective proof for this formula by showing that the sum of binomial weights satisfies a defining transition equation.

Future Directions

Quantum Non-Bumpless Pipe Dreams

- There is another combinatorial formula for double Schubert polynomial—the (non-bumpless) *pipe dream* formula due to Fomin and Kirillov.
- The pipe dreams formula for the double Schubert polynomials is genuinely different from the BPDs formula in the sense that the binomial weights are different even though they sum up to the same polynomials.
- It would be interesting to find a quantum pipe dreams formula for quantum double Schubert polynomials that generalizes these instead of bumpless pipe dreams.

Cancellation-free Formula

- Our formula is not cancellation-free. It would be interesting to make this formula cancellation free.

References

[1] Sergey Fomin, Sergei Gelfand, and Alexander Postnikov. Quantum schubert polynomials. *Journal of the American Mathematical Society*, 1997

[2] Sergey Fomin and Anatol Kirillov. The Yang-Baxter equation, symmetric functions, and Schubert polynomials. *Discrete Mathematics*, 1996.

[3] Tuong Le, Shuge Ouyang, Joseph Restivo, Leo Tao and Angelina Zhang. Quantum bumpless pipe dreams. *Forum of Mathematics, Sigma*, 2025.