Simple modules for affine type A KLR algebras via skew Specht modules

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Background

- Let $\Phi_+ = \Phi_+^{\mathrm{re}} \sqcup \Phi_+^{\mathrm{im}}$ be the positive root system of type $\mathtt{A}_{e-1}^{(1)}$, where $I = \{\alpha_0, \dots, \alpha_{e-1}\}$ is the set of simple roots, Φ_+^{re} is the set of real roots, and $\Phi_+^{\mathrm{im}} = \{d\delta \mid d \in \mathbb{Z}_{>0}\}$ is the set of imaginary roots, with $\delta = \alpha_0 + \dots + \alpha_{e-1}$ being the null root.
- For any field $\mathbb{F}$ and $\omega \in \mathbb{Z}_{\geqslant 0} I$, there is a KLR algebra R_ω over $\mathbb{F}$. These algebras categorify the positive part of the quantum group $U_q(\widehat{\mathfrak{sl}}_e(\mathbb{C}))$. The representation theory of KLR algebras is studied via *cuspidal systems*.
- Fix a convex preorder $\succcurlyeq$ on Φ_+. For $\omega \in \mathbb{Z}_{\geqslant 0} I$, a *Kostant partition* of ω is a tuple of non-negative integers $\boldsymbol{K} = (K_\beta)_{\beta \in \Psi}$ such that $\sum_{\beta \in \Psi} K_\beta \beta = \omega$. If $\beta_1 \succ \dots \succ \beta_t$ are the members of Ψ such that $K_{\beta_1} \neq 0$, then we write $\boldsymbol{K}$ in the form $\boldsymbol{K} = (\beta_1^{K_{\beta_1}} \mid \dots \mid \beta_t^{K_{\beta_t}})$. We write $\Pi(\omega)$ for the set of all *root partitions* of ω; these are pairs $\pi = (\boldsymbol{K}, \boldsymbol{\nu})$, where $\boldsymbol{K} = (\beta_1^{K_{\beta_1}} \mid \dots \mid \beta_u^{K_{\beta_u}} \mid \delta^{K_\delta} \mid \beta_{u+1}^{K_{\beta_{u+1}}} \mid \dots \mid \beta_t^{K_{\beta_t}})$ is a Kostant partition of ω, and $\boldsymbol{\nu} = (\nu^{(1)} \mid \dots \mid \nu^{(e-1)})$ is an $(e-1)$-multipartition of K_δ.

Cuspidal systems

To each $\beta \in \Phi_+^{\mathrm{re}}$, we associate a simple *cuspidal* R_β-module $L(\beta)$, and to each $(e-1)$-multipartition $\boldsymbol{\nu}$ of $d \in \mathbb{Z}_{>0}$, we associate a simple *semicuspidal* $R_{d\delta}$-module $L(\boldsymbol{\nu})$. Then, to each $\pi \in \Pi(\omega)$ as above, we associate a proper standard module

$$\bar{\Delta}(\pi) = L(\beta_1)^{\circ K_{\beta_1}} \circ \dots \circ L(\beta_u)^{\circ K_{\beta_u}} \circ L(\boldsymbol{\nu}) \circ L(\beta_{u+1})^{\circ K_{\beta_{u+1}}} \circ \dots \circ L(\beta_t)^{\circ K_{\beta_t}},$$

which has self-dual simple head $L(\pi)$. Then $\{L(\pi) \mid \pi \in \Pi(\omega)\}$ is a complete, irredundant set of simple R_ω-modules up to isomorphism and grading shift.

Main goals

In the literature, the semicuspidal modules $L(\boldsymbol{\nu})$ are not presented directly – rather, their existence is established via categorification, or they are constructed through Morita equivalences with symmetric groups and Schur algebras.

Aim

Use *skew Specht modules* to render a more direct and combinatorial-flavoured description of semicuspidal and simple R_ω-modules.

Skew Specht modules

The type $\mathtt{A}$ KLR algebra R_ω is generated by

$$\{1_{\boldsymbol{i}} \mid \boldsymbol{i} \in I^\omega\} \cup \{y_1, \dots, y_m\} \cup \{\psi_1, \dots, \psi_{m-1}\},$$

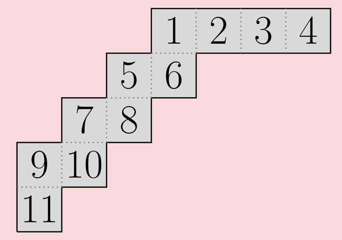
subject to a long list of relations.

We may associate residues mod e to any skew diagram, and define its *content* to be the multiset of residues it contains, with a natural correspondence between residues and simple roots.

Let $\boldsymbol{\tau}$ be a skew diagram of content ω. The tableau $\mathtt{t}^{\boldsymbol{\tau}}$ is the leading tableau – fill the nodes in order along the rows.

Example

If $\boldsymbol{\tau} = (7,4,3,2,1) \setminus (3,2,1)$, then $\mathtt{t}^{\boldsymbol{\tau}}$ is:



We define the *skew Specht module* $\mathbf{S}^{\boldsymbol{\tau}}$ to be the graded R_ω-module generated by the vector $v^{\boldsymbol{\tau}}$ in degree zero, subject to the relations:

1. $1_{\boldsymbol{i}} v^{\boldsymbol{\tau}} = \delta_{\boldsymbol{i}, \boldsymbol{i}^{\boldsymbol{\tau}}} v^{\boldsymbol{\tau}}$ for all $\boldsymbol{i} \in I^\omega$;
2. $y_r v^{\boldsymbol{\tau}} = 0$ for all $r \in [1, \mathrm{ht}(\omega)]$;
3. $\psi_r v^{\boldsymbol{\tau}} = 0$ for all $r \in [1, \mathrm{ht}(\omega) - 1]$ such that r and $r+1$ are adjacent in $\mathtt{t}^{\boldsymbol{\tau}}$;
4. $g^u v^{\boldsymbol{\tau}} = 0$ whenever $u \in \boldsymbol{\tau}$ has a node below it in $\boldsymbol{\tau}$.

The element $g^u \in R_\omega$ above is the *Garnir element*.

The skew Specht module $\mathbf{S}^{\boldsymbol{\tau}}$ has a homogeneous basis indexed by the standard $\boldsymbol{\tau}$-tableaux.

For a fixed convex preorder, we associate a tuple $\zeta(\beta)$ of skew diagrams to each root partition $\pi \in \Pi(\omega)$, working root by root.

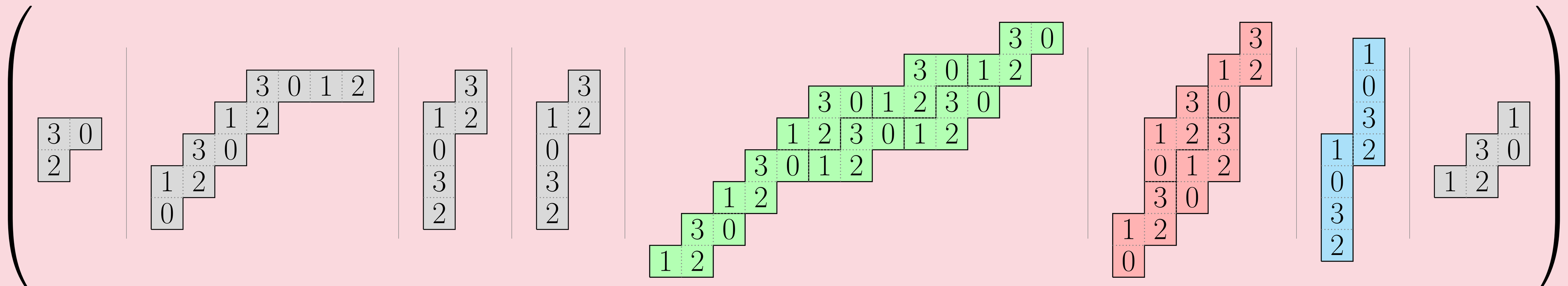
Example

Fix a certain preorder $\succcurlyeq$ on Φ_+, and let $e = 4$. Take $\pi \in \Pi(20\alpha_0 + 20\alpha_1 + 22\alpha_2 + 21\alpha_3)$ defined as

$$\pi = \left((\alpha_2 + \alpha_3 + \alpha_0 \mid 2\delta + \alpha_0 + \alpha_1 + \alpha_2 \mid (\delta + \alpha_2 + \alpha_3)^2 \mid \delta^{13} \mid \delta + \alpha_1), ((3^2,1) \mid (2^2) \mid (2)) \right).$$

Then we construct ribbons for each positive real root in the root partition π, and 'thicker' skew shapes for the tripartition of 13 appearing in π. We then concatenate these to give:

$\zeta(\pi) =$



Main result

The main result of [ADM+23] is the following.

Theorem

Let $\beta \in \Phi_+^{\mathrm{re}}$ and $K \in \mathbb{Z}_{>0}$. Up to grading shift, the real semicuspidal self-dual simple module $L(\beta^K)$ is isomorphic to the skew Specht module $\mathbf{S}^{\zeta(\beta)^K}$.

In order to describe all semicuspidal modules, we must also describe the imaginary ones $L(\boldsymbol{\nu})$. This still leaves many simple modules $L(\pi)$ that are not semicuspidal. However, two of the main results of [MNSS25] deal with both of these, as follows.

Theorem

Let $\boldsymbol{\nu}$ be an $(e-1)$-multipartition of d. Then $\mathbf{S}^{\zeta(\boldsymbol{\nu})}$ is an indecomposable semicuspidal $R_{d\delta}$-module, with simple semicuspidal head isomorphic, up to grading shift, to $L(\boldsymbol{\nu})$.

More generally, let $\pi = (\boldsymbol{K}, \boldsymbol{\nu}) \in \Pi(\omega)$. Then the skew Specht module $\mathbf{S}^{\zeta(\pi)}$ has simple head isomorphic, up to grading shift, to $L(\pi)$, and $\{\mathrm{hd}(\mathbf{S}^{\zeta(\pi)}) \mid \pi \in \Pi(\omega)\}$ gives a complete and irredundant set of simple R_ω-modules up to grading shift.

References

[ADM+23] D. Abbasian, L. Difulvio, R. Muth, G. Pasternak, I. Sholtes, and F. Sinclair, *Cuspidal ribbon tableaux in affine type A*, Algebr. Comb. **6** (2023), no. 2, 285–319.

[MNSS25] R. Muth, T. Nicewicz, L. Speyer, and L. Sutton, *A skew Specht perspective of RoCK blocks and cuspidal systems for KLR algebras in affine type A*, Represent. Theory (2025), to appear.