The number of irreducibles in the plethysm $s_\lambda[s_m]$

Ming Yean Lim

Department of Mathematics, University of Michigan

Plethysm

Plethysm is a binary operation $(f,g) \mapsto f[g]$ on the ring of symmetric functions Λ. Expressed in terms of the power sum symmetric functions p_m, plethysm is the unique operation satisfying

- for $n, m \geq 1$, $p_n[p_m] = p_{nm}$;
- for $m \geq 1$, $g \mapsto p_m[g]$ is a $\mathbb{Q}$-algebra homomorphism $\Lambda \to \Lambda$;
- for $g \in \Lambda$, $f \mapsto f[g]$ is a $\mathbb{Q}$-algebra homomorphism $\Lambda \to \Lambda$.

We are interested in the decomposition of the plethysm of Schur functions

$$s_\lambda[s_m] = \sum_{\nu \vdash nm} a^\nu_{\lambda,m} s_\nu$$

for a partition $\lambda \vdash n$. In particular, we investigate the sum

$$\sum_{\nu \vdash nm} a^\nu_{\lambda,m}. \qquad (\bigstar)$$

Wreath products

Let $\mathfrak{S}_n$ denote the symmetric group on n elements.

Let $\mathfrak{S}_m \wr \mathfrak{S}_n$ denote the *wreath product* of $\mathfrak{S}_m$ with $\mathfrak{S}_n$. We can realize it as a subgroup of $\mathfrak{S}_{nm}$ as follows:

For $1 \leq i \leq n$, define

$$\mathcal{P}_i = \{(i-1)m+1, \ldots, im\}.$$

We have inclusions $\mathfrak{S}^n_m \leq \mathfrak{S}_m \wr \mathfrak{S}_n \leq \mathfrak{S}_{nm}$, where $\mathfrak{S}^n_m$ and $\mathfrak{S}_m \wr \mathfrak{S}_n$ are identified with the stabilizers of $(\mathcal{P}_1, \ldots, \mathcal{P}_n)$ and $\{\mathcal{P}_1, \ldots, \mathcal{P}_n\}$ respectively under the natural $\mathfrak{S}_{nm}$-actions.

Representation theory of $\mathfrak{S}_n$

The irreducible characters χ^λ of $\mathfrak{S}_n$ are indexed by partitions $\lambda \vdash n$.

The *Frobenius characteristic map* gives a bridge between the ring of symmetric functions Λ and the representations of symmetric groups. The Schur function s_λ corresponds to χ^λ under this map.

All irreducible representations of $\mathfrak{S}_n$ can be realized over $\mathbb{Q}$.

Hence, the *Frobenius–Schur indicator* $\iota\chi^\lambda$ of each irreducible character χ^λ is 1:

$$\iota\chi^\lambda = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \chi^\lambda(\sigma^2) = 1.$$

Plethysm coefficients

The *plethysm coefficients* in our case of interest are given by the inner product

$$a^\nu_{\lambda,m} = \langle \chi^\nu, \operatorname{Ind}^{\mathfrak{S}_{nm}}_{\mathfrak{S}_m \wr \mathfrak{S}_n} \operatorname{Inf}^{\mathfrak{S}_m \wr \mathfrak{S}_n}_{\mathfrak{S}_n} \chi^\lambda \rangle,$$

where Ind and Inf denote the induction and inflation of characters respectively.

Sum of plethysm coefficients

Let $M(n,m)$ denote the set of $n \times n$ non-negative integer matrices with whose row and column sums are all equal to m.

We identify $\mathfrak{S}_n$ with the group of $n \times n$ permutation matrices.

Define $N^m : \mathfrak{S}_n \to \mathbb{Z}$ by

$$N^m(\sigma) = \#\{A \in M(n,m) \mid \sigma A^\mathsf{T} = A\},$$

where A^T is the transpose of A.

Theorem. *N^m is a character of $\mathfrak{S}_n$. Moreover for $\lambda \vdash n$,*

$$\langle \chi^\lambda, N^m \rangle = \sum_{\nu \vdash nm} a^\nu_{\lambda,m}.$$

Lattice points in polytopes

Let $M_n(\mathbb{R}_{\geq 0})$ denote the set of $n \times n$ matrices with entries in $\mathbb{R}_{\geq 0}$.

For $\sigma \in \mathfrak{S}_n$, define the rational convex polytope

$$\mathcal{P}(\sigma) = \{A \in M_n(\mathbb{R}_{\geq 0}) \mid A \text{ has row sums equal to 1 and } \sigma A^\mathsf{T} = A\}.$$

$N^m(\sigma)$ is the number of integer lattice points in the dialate $m\mathcal{P}(\sigma)$.

Thus, $m \mapsto N^m(\sigma)$ is an *Ehrhart quasipolynomial*.

It also follows that $(\bigstar)$ is a quasipolynomial in m.

Example

For $n = 3$, the quasipolynomials $(\bigstar)$ have been computed in [1].

For $n = 6$ and $\lambda = 6$, we compute using SageMath and the above theorem that

$$\sum_{\nu \vdash 6m} a^\nu_{6,m} = \frac{243653}{14347055923200000}m^{15} + \frac{243653}{31882346496000}m^{14} + \frac{91173671}{573882236928000}m^{13} + \frac{5954623}{2942985830400}m^{12} + \frac{3895930519}{220723937280000}m^{11} + \frac{149644967}{1337720832000}m^{10} + \frac{1072677673}{2006581248000}m^{9} + \frac{14723521}{7431782400}m^{8} + \frac{350041981}{59719680000}m^{7} + O(m^6).$$

Proof sketch

Define the function $\theta : \mathfrak{S}_{nm} \to \mathbb{C}$ by $\theta(\sigma) = \#\{\tau \in \mathfrak{S}_{nm} \mid \tau^2 = \sigma\}$. Then $\langle \chi^\nu, \theta \rangle = \iota\chi^\nu = 1$, so $\theta = \sum_{\nu \vdash nm} \chi^\nu$, and

$$\sum_{\nu \vdash nm} a^\nu_{\lambda,m} = \langle \theta, \operatorname{Ind}^{\mathfrak{S}_{nm}}_{\mathfrak{S}_m \wr \mathfrak{S}_n} \operatorname{Inf}^{\mathfrak{S}_m \wr \mathfrak{S}_n}_{\mathfrak{S}_n} \chi^\lambda \rangle.$$

By Frobenius reciprocity and a bit of manipulation,

$$\langle \theta, \operatorname{Ind}^{\mathfrak{S}_{nm}}_{\mathfrak{S}_m \wr \mathfrak{S}_n} \operatorname{Inf}^{\mathfrak{S}_m \wr \mathfrak{S}_n}_{\mathfrak{S}_n} \chi^\lambda \rangle = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \left(\frac{1}{m!^n} \#\{\tau \in \mathfrak{S}_{nm} \mid \tau^2\sigma^{-1} \in \mathfrak{S}^n_m\} \right) \chi^\lambda(\sigma).$$

The proof is completed by constructing a surjective $m!^n$-to-1 map

$$\{\tau \in \mathfrak{S}_{nm} \mid \tau^2\sigma^{-1} \in \mathfrak{S}^n_m\} \to \{A \in M(n,m) \mid \sigma A^\mathsf{T} = A\}.$$

Permutation equivalence

Call two matrices $A, B \in M(n,m)$ *permutation equivalent* and write $A \sim B$ if A can be transformed into B by row and column permutations.

Let $T(n,m) = \{A \in M(n,m) \mid A \sim A^\mathsf{T}\}$ denote the subset of matrices in $M(n,m)$ that are permutation equivalent to their transpose.

Theorem. *In the case $\lambda = n$,*

$$\sum_{\nu \vdash nm} a^\nu_{n,m} = \langle 1, N^m \rangle_{\mathfrak{S}_n} = \#T(n,m)/\sim.$$

Example

We compute that $s_3[s_3] = s_9 + s_{72} + s_{63} + s_{522} + s_{441}$.

Correspondingly, there are five elements of $T(3,3)/\sim$, represented by

$$\begin{pmatrix} 3&0&0\\0&3&0\\0&0&3 \end{pmatrix}, \begin{pmatrix} 3&0&0\\0&2&1\\0&1&2 \end{pmatrix}, \begin{pmatrix} 2&1&0\\1&1&1\\0&1&2 \end{pmatrix}, \begin{pmatrix} 2&1&0\\1&0&2\\0&2&1 \end{pmatrix}, \begin{pmatrix} 1&1&1\\1&1&1\\1&1&1 \end{pmatrix}.$$

Assigning matrices to partitions

Question. *Can we define a function $\phi_{n,m} : T(n,m)/\sim \to \{\nu \vdash nm\}$ such that $a^\nu_{n,m} = \#(\phi_{n,m})^{-1}(\nu)$?*

An analogous problem

Young's rule implies that

$$(s_m)^n = \sum_{\nu \vdash nm} K_{\nu,m^n} s_\nu$$

where $K_{\lambda,\mu}$ denote the *Kostka numbers*.

The *RSK algorithm* can be used to assign a partition $\nu \vdash nm$ to each $n \times n$ non-negative integer symmetric matrix A with row and column sums m. Namely, we apply the RSK algorithm to A and take ν to be the shape of either of the resulting tableaux.

This shows that

$$\sum_{\nu \vdash nm} K_{\nu,m^n} = N^m(1_{\mathfrak{S}_n}).$$

References

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mylim@umich.edu