Degenerating brick manifolds and cubulating the associahedron

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The associahedron

The n-dimensional associahedron has vertices correspond to triangulations of an n-gon which are connected by an edge when they only differ in the placement of one diagonal. There are multiple realizations of the associahedron as a lattice polytope, but Loday's realization may be the most famous.

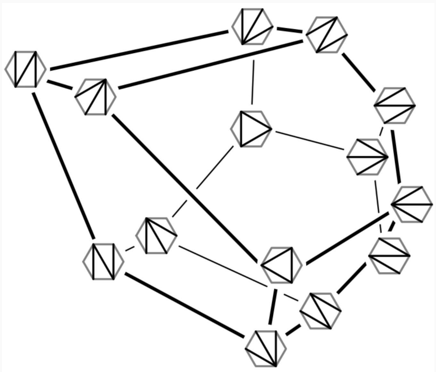


Fig. 1: Image of Loday's associahedron from [PS12]

Toric Varieties

Definition

- A **torus** is an algebraic group which is isomorphic to the group $(\mathbb{C}^\times)^n = (\mathbb{C} \setminus \{0\})^n$ for some n.
- A normal variety with an algebraic action of some torus T is called a **toric variety** if it has a dense T orbit.

Given a lattice polytope $P \subseteq \mathbb{R}^n$, we can construct a corresponding projective toric variety

$$X_P := \operatorname{Proj} \mathbb{C}\left[\overline{\mathbb{R}_+(P \times \{1\})} \cap \mathbb{Z}^{n+1}\right]$$

with proj grading given by the last coordinate.
P and X_P encode exactly the same information, so there is a convex geometry to algebraic geometry dictionary:

Convex Geometry	Algebraic Geometry
Polyhedral fan	abstract toric variety
Polytope P	$X_P \hookrightarrow \mathbb{P}^n$ toric variety
$\dim_{\mathbb{R}}$	$\dim_{\mathbb{C}}$
vertices	T-fixed points
# lattice points in kP	$\dim H^0(X_P, \mathcal{O}_{X_P}(k))$
simple	rationally smooth
regular polyhedral subdivision	Gröbner degeneration

Every algebraic action of $T = (\mathbb{C}^\times)^k$ on $\mathbb{P}^n$ is of the form

$$(t_1, \dots, t_k) \cdot (x_1 : \dots : x_m) = (x_1 \prod_{i=1}^k t_i^{w_{1,i}} : \dots : x_m \prod_{i=1}^k t_i^{w_{m,i}})$$

with $w_{j,i} \in \mathbb{Z}$. A T-fixed point p of a T-equivariantly embedded variety $\iota\colon X \hookrightarrow \mathbb{P}^n$ such that $\iota(p) = (0 : \dots : 0 : x_j : 0 : \dots : 0) \in \mathbb{P}^n$ corresponds to a vertex $(w_{j,1}, w_{j,2}, \dots, w_{j,k}) \in \mathbb{Z}^k$. The polytope associated to X is the convex hull of all such vertices.

Gröbner degenerations

A Gröbner degeneration of an affine variety amounts to replacing $V(I)$ with $V(\text{init}_{\preceq}(I))$ for some monomial ordering $\preceq$. Moment polytopes of the irreducible components of the degeneration subdivide the original moment polytope and all 'regular' polyhedral subdivisions arise in this manner [Stu91].

Toric variety of the associahedron

Definition

- The **Grassmannian** $\mathrm{Gr}_{k,n}$ is a projective variety whose points correspond to k-dimensional vector subspaces of $\mathbb{C}^n$
- The (full) **flag variety** Fl_n is a projective variety whose points correspond to full flags $\{0\} = F_0 < F_1 < \dots < F_{n-1} < F_n = \mathbb{C}^n$ where F_i is a vector space of dimension i

For $Q = q_1 \cdots q_k$ a word in the alphabet $\{1, \dots, n-1\}$, $\mathrm{Brick}^Q \subseteq \mathrm{Gr}_{q_1,n} \times \mathrm{Gr}_{q_2,n} \times \dots \times \mathrm{Gr}_{q_k,n} \hookrightarrow \mathrm{Fl}_n^k$.
Viewed as a subset of Fl_n^k, the brick manifold Brick^Q consists of all sequences of flags $(F_1, F_2, \dots, F_k) \in \mathrm{Fl}_n^k$ such that F_{i-1} and F_i differ only in their q_i dimensional part, F_k is required to be the flag with i-dimensional part equal to $\langle \mathbf{e_n}, \mathbf{e_{n-1}}, \dots, \mathbf{e_{n-i+1}} \rangle$, and F_0 denotes the flag with i-th dimensional part equal to $\langle \mathbf{e_1}, \dots, \mathbf{e_i} \rangle$.

Theorem [Esc16]

The toric variety of Loday's realization of the n-dimensional associahedron is given by Brick^Q for $Q = 1, 2, \dots, n-1, 1, 2, \dots, n-1, 1, 2, \dots, n-2, \dots, 1, 2, 3, 1, 2, 1$.

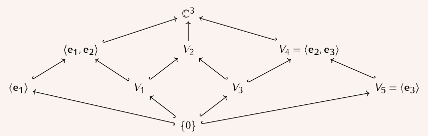


Fig. 2: The space of (V_1, V_2, V_3) satisfying the above inclusions is Brick^{12121}. It is the toric variety of Loday's realization of the 2D associahedron

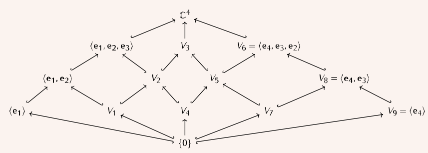


Fig. 3: $\mathrm{Brick}^{123123121}$ is the toric variety of the 3D associahedron and is constructed as the moduli space of $(V_1, V_2, \dots, V_9)$ satisfying the inclusions of the above magyar diagram.

Considering the flags $V_1 \subseteq V_2 \subseteq V_3$, $V_4 \subseteq V_5 \subseteq V_6$, and $V_7 \subseteq V_8 \subseteq V_6$ in Fig. 3 affords a useful embedding of $\mathrm{Brick}^{123123121}$ into Fl_4^3

Torus fixed points

Let $T \cong (\mathbb{C}^\times)^n$ be the group of $n \times n$ diagonal matrices with nonzero diagonal entries. The action of T on $\mathbb{C}^n$ extends to an action on $\mathrm{Gr}_{k,n}$ by $t \cdot V = \{t \cdot \mathbf{x} : \mathbf{x} \in V\}$ and then on Fl_n by $(t \cdot F_\bullet)_i = t \cdot F_i$.
Linear subspaces $V \leq \mathbb{C}^n$ such that $t \cdot V = V$ for all $t \in T^n$ are exactly the coordinate subspace $V = \mathrm{Span}\{e_{i_1}, \dots, e_{i_k}\}$.

$$|\mathrm{Gr}_{k,n}^{T^n}| = \binom{n}{k}$$

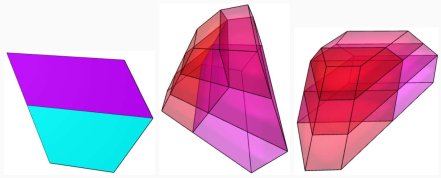
Flags F on $\mathbb{C}^n$ such that $t \cdot F = F$ for all $t \in T$ are exactly flags such that each F_i is a coordinate subspace. $|\mathrm{Fl}_n^T| = n!$
A point $(V_1, \dots, V_k)$ of the brick variety is a torus fixed point iff each V_i is a coordinate subspace.

For Brick^{12121} the fixed points are:

V_1	V_2	V_3	V_4	V_5	Corresponding vertex
$\langle e_1 \rangle$	$\langle e_1, e_2 \rangle$	$\langle e_2 \rangle$	$\langle e_2, e_3 \rangle$	$\langle e_3 \rangle$	(3, 4, 1)
$\langle e_1 \rangle$	$\langle e_1, e_3 \rangle$	$\langle e_3 \rangle$	$\langle e_2, e_3 \rangle$	$\langle e_3 \rangle$	(3, 2, 2)
$\langle e_2 \rangle$	$\langle e_1, e_2 \rangle$	$\langle e_2 \rangle$	$\langle e_2, e_3 \rangle$	$\langle e_3 \rangle$	(2, 5, 1)
$\langle e_2 \rangle$	$\langle e_2, e_3 \rangle$	$\langle e_2 \rangle$	$\langle e_2, e_3 \rangle$	$\langle e_3 \rangle$	(1, 5, 2)
$\langle e_2 \rangle$	$\langle e_2, e_3 \rangle$	$\langle e_3 \rangle$	$\langle e_2, e_3 \rangle$	$\langle e_3 \rangle$	(1, 4, 3)

A cubulation of the associahedron

Using Escobar's construction of the brick variety below and the 'orbit degeneration' of [KMS06], we get a subdivision of the n-dimensional associahedron into pieces which are combinatorially cubes. The subdivision has $n!$ cubes and $(n+1)!$ vertices.



Bruhat order and Bruhat Interval Polytopes

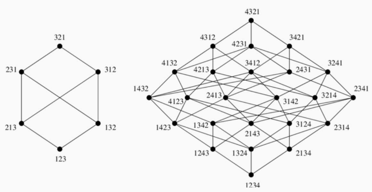


Image credit: Adam Hammett and Boris Pittel

Definition

- The **simple reflection** s_i transposes i and $i+1$
- A **word** for a permutation π is a sequence of simple reflections $q_1, q_2, \dots, q_k$ so that $\pi = q_1 \cdot q_2 \cdots q_k$. A **reduced word** for π is a word for π with minimal length $\ell(\pi)$
- $\pi \leq \tau$ in Bruhat order if every reduced word for τ contains a reduced word for π as a subword

The greatest element of Bruhat order is $w_0 = n \quad n-1 \quad \dots \quad 1$

Definition [BEW24; KW15]

For $u, v \in S_n$, the twisted Bruhat interval polytope $Q_{u,v}$ is the convex hull of $(n+1-w^{-1}(1), n+1-w^{-1}(2), \dots, n+1-w^{-1}(n))$ for $u \leq w \leq v$

Theorem [LMP21]

If $uv^{-1} = s_k s_{k-1} \cdots s_{k-r}$ then $Q_{u,v}$ is combinatorially a cube

Results

Let $a_i = s_1 s_2 \dots s_{n-i+1} \in S_n$ for $2 \leq i \leq n$ and $a_1 = a_2$

Definition

The **Minkowski sum** of sets is $S_1 + S_2 := \{a + b : a \in S_1, b \in S_2\}$

Theorem [Gandhi-U]

Loday's realization of the $n-1$ dimensional associahedron is equal (up to translation) to the Minkowski decomposition $\sum_{j=1}^n Q_{a_j, s_j s_{j+1} \dots s_{n-1} a_j}$ where $a_j = \prod_{i=2}^j a_i$.
Additionally it has a mixed subdivision into $(n-1)!$ cubes given by $\bigcup_{u_\bullet} \sum_{i=1}^n Q_{u_i a_i, u_{i+1}}$ where the union is taken over sequences $u_\bullet \in (S_n)^n$ where $u_1 = id$, $u_{i+1} \leq u_i a_i$, $\ell(u_i a_i) = \ell(u_i) + \ell(a_i)$, and $u_{n+1} = w_0$

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