TYPE C K–STANLEY SYMMETRIC FUNCTIONS AND KRAŚKIEWICZ–HECKE INSERTION

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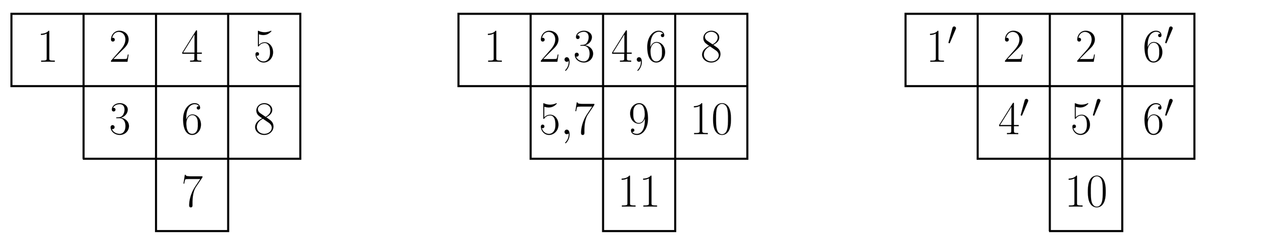
Signed Permutations

- Permutations of $[\overline{n}] \cup [n]$ such that $w(i) = -w(\bar{i})$.
- Fully determined by $w([n])$.
- Generated by $s_0 = (\bar{1}, 1)$ and $s_i = (\overline{i+1}, \bar{i})(i, i+1)$ for $i > 0$.

Hecke Words

- Represents series of generators applied to create a permutation
- Generator is only applied if it increases the length of the permutation
- Reduced word if minimal
- Follows the rules: $s_i \circ s_i = s_i$; $s_i \circ s_j = s_j \circ s_i$ if $|i-j| > 1$; $s_i \circ s_{i+1} \circ s_i = s_{i+1} \circ s_i \circ s_{i+1}$ if $i > 0$; $s_0 \circ s_1 \circ s_0 \circ s_1 = s_1 \circ s_0 \circ s_1 \circ s_0$.

Shifted Tableaux



Standard (ShSYT) **Set-Valued Standard** (ShSet) and **Semistandard** (ShSVT)

Related Symmetric Functions

- GQ **functions**, introduced in [1], indexed by shifted partitions:
  - K–theory representatives for the Lagrangian Grassmannian
  - combinatorial formula in terms of shifted set-valued tableaux
- **Type C K–Stanley symmetric function**, introduced in [2], indexed by signed permutations
  - stabilization of Type C Schubert polynomial
  - combinatorial formula in terms of parenthesizations
  $$(a_1, \ldots, a_k | a_{k+1}, \ldots, a_m) \ldots (a_n, \ldots, a_p \mid a_{p+1} \ldots a_q)$$
  where $a_1 \circ \cdots \circ a_q = w$ and $a_1 > \cdots > a_k \neq a_{k+1} < \cdots < a_m$ with equality allowed when $0 = a_k = a_{k+1}$ (and likewise for each other parenthesization).

Selected References

[1] T. Ikeda and H. Naruse. *K-theoretic analogues of factorial Schur P-and Q-functions* Adv. in Math. (243) 22–66 2013.

[2] A. Kirillov and H. Naruse. *Construction of double Grothendieck polynomials of classical types using idCoxeter algebras* Tokyo J. of Math. (39) 695–728 2017.

[3] W. Kraśkiewicz. *Reduced decompositions in hyperoctahedral groups* C.R.M. (309) 903–907 1989.

[4] J. Stembridge. *Some combinatorial aspects of reduced words in finite Coxeter groups* Trans. AMS (349) 1285–1332 1997.

[5] H. Tamvakis. *Tableau formulas for skew Grothendieck polynomials* J. M.S.Japan (1) 1–26 2023.

Main Theorem

Kraśkiewicz–Hecke insertion KH is a bijection:

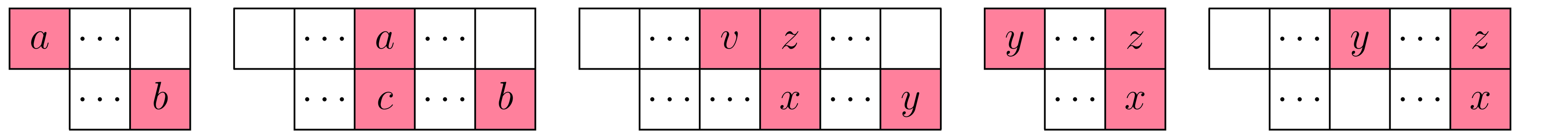
$$KH : \mathbb{N}^n \xrightarrow{\sim} \bigsqcup_{\lambda \vdash m \leq n \text{ strict}} \mathsf{SDT}(\lambda) \times \mathsf{ShSet}_n(\lambda).$$

Moreover, for $KH(\mathbf{a}) = (P, Q)$, the words $\mathbf{a}$ and the row reading word $\rho(P)$ are Hecke words for the same signed permutation.

Strict Decomposition Tableau

A **Strict Decomposition Tableau** (SDT) is a shifted tableau with nonnegative integer entries so:

- Each row is a unimodal sequence
- Every entry of row R_{i+1} is less than the first entry of row R_i
- For $a \leq b < c$, $x < y \leq z$, and $v < z$ the following configurations are avoided:

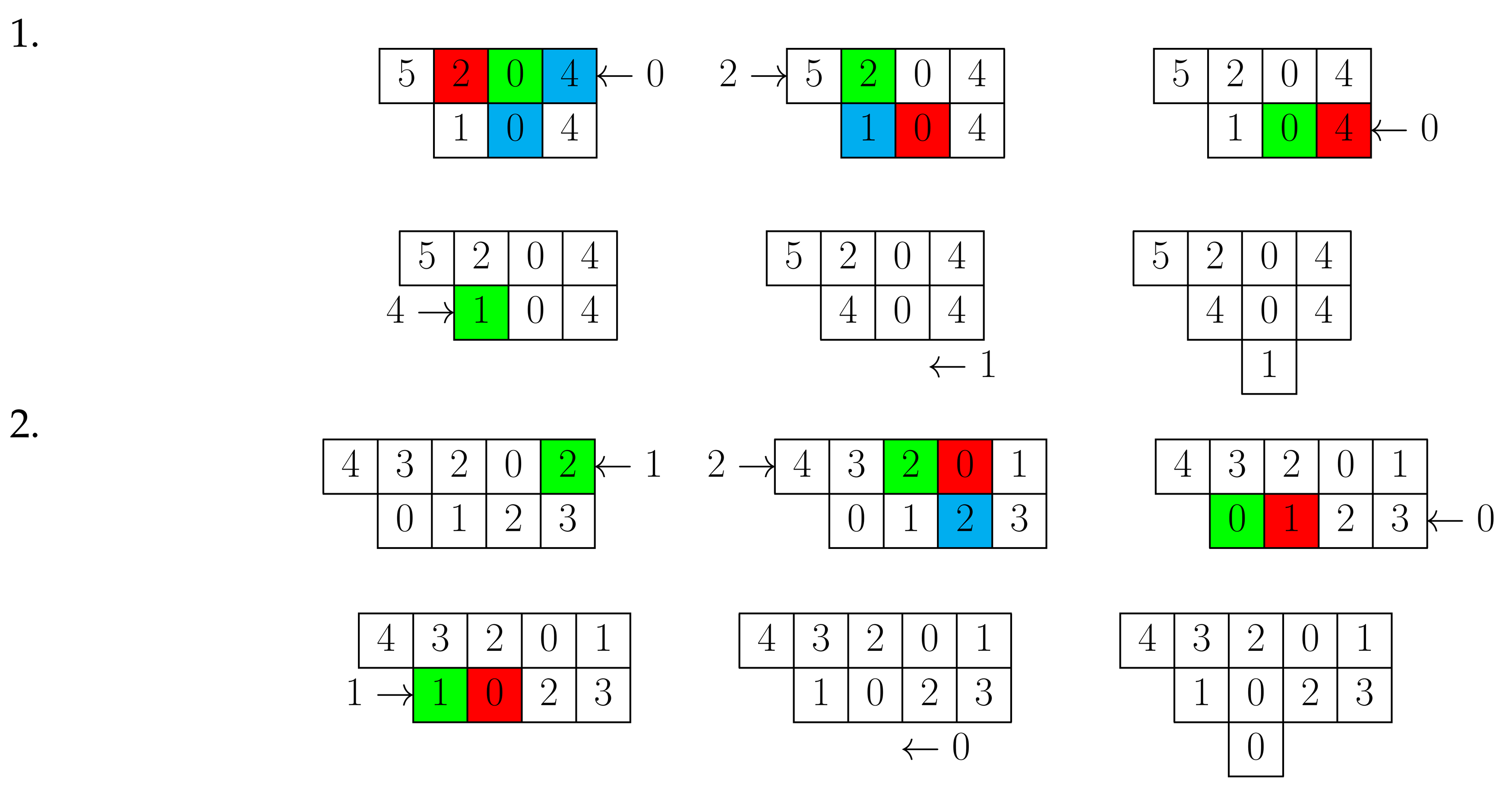


See Insertion Example for examples.

Kraśkiewicz-Hecke Insertion

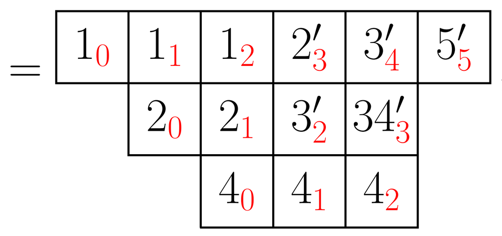
- Is comprised of two steps: right insertion into the increasing part of the row and left insertion into the decreasing part of the row.
- The recording tableau tracks where each insertion step terminates, which could be an already existing box.
- Performing right or left insertion can require comparing entries in both the row inserted into and the row below, with possible outputs split into three cases each.
- Restricts to Kraśkiewicz insertion [4] for reduced words of signed permutations, and differs only when inserting an element already in the corresponding part of the row

Insertion Examples



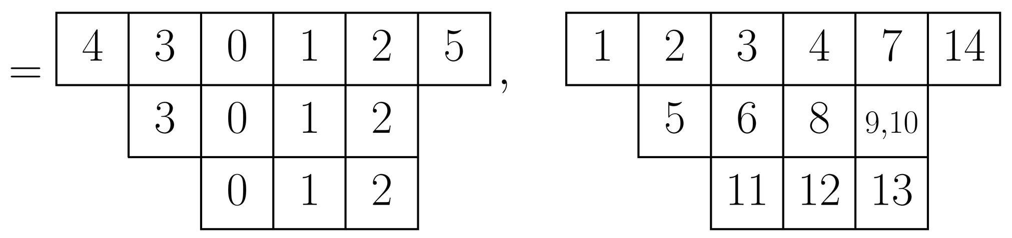
Top Fully Commutative

A signed permutation is **fully commutative** if its reduced words contain no braid relations and **top** if its reduced words also avoid the consecutive subword $(1, 0, 1)$ – for example the permutation with the reduced word $(1, 4, 0, 1)$ is not top. Stembridge showed reduced words for top fully commutative permutations are in bijection with shifted standard tableaux [4]. Our map res extends this to a bijection from from $\mathsf{ShSVT}_p(\lambda/\mu)$ to parenthesizations of a top signed permutation $w^{\lambda/\mu}$ (an equivalent map appears in [5]).

$T =$  , $\mathsf{res}(T) = (|0, 1, 2)(3|0, 1)(4, 2|3)(3|0, 1, 2)(5|)$

Corollary 1 *For $\mu \subseteq \lambda$ strict shapes, $G^C_{w(\lambda/\mu)} = GQ_{\lambda/\mu}$.*

Theorem 1 *For $T' \in \mathsf{ShSet}(\lambda)$, $KH(\mathsf{res}(T')) = (P^\lambda, T')$.*

$KH(\mathsf{res}(T')) =$ 

Conjectures

Conjecture 1 *For w a signed permutation, there are non-negative integers $a^C_w(\lambda)$ so*

$$G^C_w = \sum_{\lambda \text{ strict}} (-1)^{|\lambda| - \ell(w)} a^C_w(\lambda) \cdot GQ_\lambda.$$

where $a^C_w(\lambda) = \#\{P \in \mathsf{SDT}(\lambda) : \rho(P) \text{ is a Hecke word for } w\}$.

Combined with Corollary 1, this would show

$$GQ_{\lambda/\mu} = \sum_\nu a^C_{w^{\lambda/\mu}}(\nu) \cdot GQ_\nu,$$

answering questions from work by Lewis-Marberg and Marberg.

Verified for Hecke words up to size 11 with letters from $\{s_0, s_1, \ldots, s_6\}$.

Major Obstacle