# Log-concavity and log-convexity via distributive lattices

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## Log-concavity, log-convexity, and distributive lattices

Consider a sequence of real numbers $(a_n) = (a_n)_{n\geq 0} = a_0, a_1, a_2, \ldots$.
Say that $(a_n)$ is **log-concave** if, for all $n \geq 1$, we have

$$a_n^2 \geq a_{n-1}a_{n+1}.$$

Say that $(a_n)$ is **log-convex** if, for all $n \geq 1$, we have

$$a_n^2 \leq a_{n-1}a_{n+1}.$$

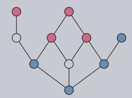
Let $P = (P, \preceq)$ be a poset (partially ordered set).
A **lower (order) ideal** of $P$ is $I \subseteq P$ such that

$$x \in I \text{ and } y \preceq x \text{ implies } y \in I.$$

An **upper (order) ideal** of $P$ is $J \subseteq P$ such that

$$x \in J \text{ and } y \succeq x \text{ implies } y \in J.$$

**Example of Ideals**



Upper ideal (red), lower ideal (blue)

Say that poset $L$ is a **distributive lattice** if every pair $x, y \in L$ has a greatest lower bound or **meet**, $x \wedge y$, as well as a least upper bound or **join**, $x \vee y$, and it satisfies either of the two equivalent distributive laws. Our main tool follows from the FKG inequality [FKG71].

## New tool: Order Ideal Lemma

Let $L$ be a distributive lattice and suppose that $I, J \subseteq L$ are ideals.

(a) If $I, J$ are both lower ideals or both upper ideals then

$$|I| \cdot |J| \leq |I \cap J| \cdot |L|.$$

(b) If one of $I, J$ is a lower ideal and the other is upper then

$$|I| \cdot |J| \geq |I \cap J| \cdot |L|.$$

## Strategy

Our general strategy for proving log-convexity of a sequence $(a_n)_{n\geq 0}$:

- Construct distributive lattices $L_n$ with $|L_n| = a_n$;
- Find inside $L_{n+1}$ two lower order ideals $I, J$ such that $|I| = |J| = a_n$ and $|I \cap J| = a_{n-1}$.

Then we will be done by part (a) of the Order Ideal Lemma. Similarly, part (b) can be used to prove log-concavity.

## Application 1: Catalan numbers

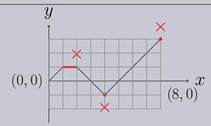
The ubiquitous **Catalan numbers** can be explicitly given as

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

A **Dyck path of semilength** $n$ is a lattice path $P$ satisfying:

1. $P$ starts at $(0,0)$ and ends at $(2n, 0)$;
2. $P$ uses up steps $U$ parallel to $[1,1]$ and down steps $D$ parallel to $[1,-1]$;
3. never goes below the $x$-axis.

**Non-example**



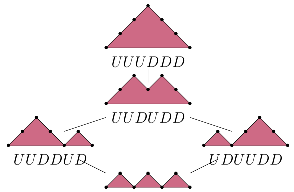




Figure 1. The poset $\mathcal{D}_3$

Let $\mathcal{D}_n$ be the set of all Dyck paths of semilength $n$. It is well known that $C_n = |\mathcal{D}_n|$. If $P \in \mathcal{D}_n$, then let $A(P)$ be the physical area enclosed by $P$ and the $x$-axis. Turn $\mathcal{D}_n$ into a poset by letting

$$P \preceq Q \iff A(P) \subseteq A(Q).$$

See Figure 1 for the poset $\mathcal{D}_3$. For all $n \geq 0$ we have that $\mathcal{D}_n$ is a distributive lattice [FP05].

**Theorem. The sequence $(C_n)$ is log-convex.**

**Proof (L-Sagan).** Let $L = \mathcal{D}_{n+1}$ so that $|L| = C_{n+1}$. The maximal element in $\mathcal{D}_n$ is the path $U^nD^n$. Construct the lower ideals

$$I = \{P \in L \mid P \preceq U^nD^nUD\}, \quad J = \{P \in L \mid P \preceq UDU^nD^n\}.$$

Therefore $I \cong J \cong \mathcal{D}_n$ so that $|I| = |J| = C_n$. Also

$$I \cap J = \{P \in L \mid P \preceq UDU^{n-1}D^{n-1}UD\}.$$

Therefore $I \cap J \cong \mathcal{D}_{n-1}$ so that $|I \cap J| = C_{n-1}$. Thus, by the Order Ideal Lemma

$$C_n^2 = |I| \cdot |J| \leq |I \cap J| \cdot |L| = C_{n-1}C_{n+1}$$

as desired. □

## Application 2: Order polynomials

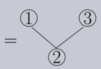
For a positive integer $p$ we let $[p] = \{1, 2, \ldots, p\}$. Let $(P, \preceq)$ be a poset on $[p]$. A **$P$-partition with range** $[n]$ is a map $f : P \to [n]$ such that for all $x \prec y$:

1. $f(x) \geq f(y)$, i.e., $f$ is order reversing, and
2. If $x > y$ then $f(x) > f(y)$.

Let

$$\mathcal{O}_P(n) = \{f \mid f \text{ is a } P\text{-partition with range } [n]\}.$$

**Example of $P$-partitions.** If $p = 3$ and

$P =$ 


$$\therefore \mathcal{O}_P(n) = \{f : P \to [n] \mid f(2) \geq f(3) \text{ and } f(2) > f(1)\}.$$

The **order polynomial** of $P$ is

$$\Omega_P(n) = |\mathcal{O}_P(n)|.$$

Turn $\mathcal{O}_P(n)$ into a poset by letting

$$f \preceq g \iff f(x) \leq g(x) \text{ for all } x \in P.$$

See Figure 2 for an example of $\mathcal{O}_P(3)$ with $P$ in the previous example.

The resulting poset is in fact a distributive lattice. Together with Order Ideal Lemma, we prove:

**Theorem (L-Sagan). For any $P$ on $[p]$, the sequence $(\Omega_P(n))_{n\geq 1}$ is log-concave.**

- This result was proved in the special case of naturally labeled $P$ by Chan, Pak, and Panova [CPP23]. It also leads to the log-concavity of $(s_\lambda(1^n))_{n\geq 0}$, a sequence of specializations of Schur functions.

**Example.**

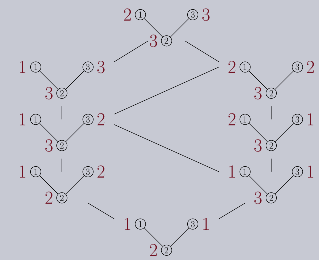
$\mathcal{O}_P(3) =$ 

Figure 2. The poset $\mathcal{O}_P(3)$

## Other results

- Call $a_0, a_1, a_2, \ldots$ **log-concave at** $n$ if $a_n^2 \geq a_{n-1}a_{n+1}$. Similarly define being **log-convex at** $n$. A sequence $(\ell_n)$ of real numbers is a **generalized Lucas sequence** if it satisfies the recursion

$$\ell_n = \ell_{n-1} + \ell_{n-2}$$

  for $n \geq 2$. Using the Order Ideal Lemma we prove that the sequence $(\ell_n)$ is log-concave at odd indices and log-convex at even ones.
- The **Stirling numbers of the second kind** are

$$S(n,k) = \text{number of partitions of } [n] \text{ into } k \text{ subsets (blocks)}.$$

  By defining a new poset on such partitions which is a distributive lattice, we have proved that for fixed $k$, the sequence $(S(n,k))_{n\geq 0}$ is log-concave.

## Challange Conjecture

Recall that the **signless Stirling numbers of the first kind** are

$$c(n,k) = \#\{\pi \in \mathfrak{S}_n \mid \pi \text{ has } k \text{ cycles in its disjoint cycle decomposition}\}.$$

We have checked the following conjecture for $1 \leq k \leq n \leq 100$.

**Conjecture**: Given $k$, there is an integer $N_k$ such that $(c(n,k))_{n\geq 0}$ is log-concave for $n < N_k$ and log-convex for $n \geq N_k$.

Full details are available in [LS].

## References

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