Minkowski sums of alcoved polytopes

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Alcoved polytopes

A polytope in $\mathcal{H}_n = \{x_1 + \cdots + x_n = 0\} \subset \mathbb{R}^n$ is **alcoved** if all its facet normals are parallel to the roots $e_i - e_j$ for some $i \neq j \in [n]$. Equivalently, a polytope is alcoved if it is determined by the parameters $a_{i,j} \in \mathbb{R}$ for $1 \leq i, j, \leq n$ via the equation $x_1 + \cdots + x_n = 0$ and the inequalities

$$x_i - x_j \leq a_{i,j} \text{ for all } i, j \in [n], i \neq j. \tag{1}$$

Unlike some other families of polytopes alcoved polytopes are not closed under Minkowski sums in general. This naturally raises the question when alcoved polytopes add.

Main question

How to characterize pairs of alcoved polytopes $P, Q \subseteq \mathcal{H}_n$, such that their Minkowski sum $P + Q$ is alcoved?

Some motivation

- The cone of alcoved polytopes (given by triangle inequolities in $a_{i,j}$) has a natural fan structure given by combinatorial alcoved polytopes. This is called **type fan of alcoved polytopes**. Understanding combinatorics of type fan is equivalent to understanding the compatibility of alcoved polytopes.
- **Binary geometries** are affine varieties with stratifications determined by certain simplicial complexes. Classical example of binary geometry come from a presentation of the associahedron as a Minkowski sum of symplices and a more recent one come from analogous presentaion for pellytopes. In both cases, all polytopes involved are alcoved.
- Certain **scattering amplitudes** may be presented as $\varepsilon \to 0$ limit of integrals of the following form called *stringy* integrals:

$$\int_{\mathbb{R}^d_{>0}} \frac{dy}{y} \prod_{j=1}^{d} x_j^{s_j} \prod_f f(y)^{\varepsilon s_f},$$

where $f(y)$ are some given irreducible polynomials. In the case when the Minkowski sum of the Newton polytopes of f is the ABHY associahedron, it produces the classical Koba-Nielsen string integral. We are interested in more general alcoved polytopes.

Flag property for type fan

Our first result shows that the type fan satisfy a flag property. In particular, this implies that to understand combinatorics of type fan it is enough to understand which pairs of alcoved polytopes are compatible.

Theorem (Nick Early, Lukas Kühne, LM)

Let $P_1, \ldots, P_k$ be alcoved polytopes in $\mathcal{H}_n$. Suppose P_i and P_j are pairwise compatible for all $i \neq j \in [n]$. Then the entire collection is compatible, i.e., $P_1 + \cdots + P_k$ is alcoved.

Alcoved simplices

An **ordered set partition** of the set $[n]$ is an ordered tuple $\mathbf{S} = (B_1, \ldots, B_\ell)$ of pairwise disjoint subsets $B_i \subseteq [n]$ with $\cup_{j=1}^{\ell} B_j = [n]$.

To each ordered set partition $\mathbf{S} = (B_1, \ldots, B_\ell)$ of $[n]$ we associate an *alcoved simplex* $\Delta_\mathbf{S}$ in the hyperplane $\mathcal{H}_n$ defined by the following set of (in)equalities:

$$\begin{aligned} x_i &= x_j && \text{for every } i, j \in B_k \text{ and every } 1 \leq k \leq \ell, \\ x_i &\geq x_j && \text{for every } i \in B_k,\ j \in B_{k+1} \text{ and every } 1 \leq k \leq \ell - 1, \\ x_i &\geq x_j - 1 && \text{for every } i \in B_\ell,\ j \in B_1. \end{aligned}$$

Theorem

Every alcoved simplex in $\mathcal{H}_n$ is equal to $\Delta_\mathbf{S}$ for some ordered set partition $\mathbf{S}$ up to shift and dilation.

We will encode combinatorics of $\Delta_\mathbf{S}$ in a graph a graph $G_\S$ as a partially directed graph on n vertices which has

- an undirected clique on the set B_i;
- directed edge $b_i \to b_{i+1}$ for $1 \leq i \leq \ell$ (regarded cyclically) where $b_j \in B_j$ is the smallest element of a block B_j.

Example

The alcoved simplex $\Delta_{(1,2\,3,4)}$ in $\mathbb{R}^4$ of the ordered set partition $(1, 2\,3, 4)$ is defined by $x_1 + \cdots + x_4 = 0$ and the (in)equalities

$$x_1 \geq x_2 = x_3 \geq x_4 \geq x_1 - 1.$$

Its vertices are $(0,0,0,0)$, $\left(\frac{3}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}\right)$ and $\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, -\frac{3}{4}\right)$. The graph $G_{(1,2\,3,4)}$ is depicted below.

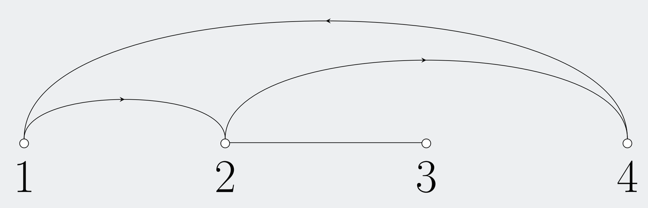

Figure 1. The graph $G_\mathbf{S}$ of the ordered set partition $\mathbf{S} = (1, 2\,3, 4)$.

Compatibility of alcoved simplices

Let us denote by $G_{\mathbf{S},\mathbf{T}}$ to be the union $G_\mathbf{S} \cup G_\mathbf{T}^{op}$. We call edges in $G_\mathbf{S}$ **upper** and those in $G_\mathbf{T}^{op}$ **lower**.

Let C be a cycle in $G_{\mathbf{S},\mathbf{T}}$. An **upper path segment** of C is a collection of consecutive upper edges in C. We call a cycle **violating** if it has at least two disjoint upper path segments and visits every vertex of $G_{\mathbf{S},\mathbf{T}}$ at most once.

Theorem (Nick Early, Lukas Kühne, LM)

The ordered set partitions $\mathbf{S}, \mathbf{T}$ on $[n]$ are compatible if and only if $G_{\mathbf{S},\mathbf{T}}$ does not have a violating cycle.

Interlacing and compatibility

We call order set partitions $\mathbf{S}, \mathbf{T}$ *4-interlaced* if there exist 4 distinct elements $a, b, c, d \in [n]$ such that the ordered set partitions of $\mathbf{S}$ and $\mathbf{T}$ restrict to respectively

$$\mathbf{S}|_{a,b,c,d} = (a, b, c, d) \quad \text{and} \quad \mathbf{T}|_{a,b,c,d} = (c, b, a, d).$$

We say that $\mathbf{S}$ and $\mathbf{T}$ are *6-interlaced* if there exist 6 distinct elements $a, b, c, d, e, f \in [n]$ such that the ordered set partitions of $\mathbf{S}$ and $\mathbf{T}$ restricts respectively to one of the two pairs

$$\mathbf{S}|_{a,b,c,d,e,f} = (a, b, c, d, e, f) \quad \text{and} \quad \mathbf{T}|_{a,b,c,d,e,f} = (c, d, a, b, e, f).$$
$$\mathbf{S}|_{a,b,c,d,e,f} = (a, b, c, d, e, f) \quad \text{and} \quad \mathbf{T}|_{a,b,c,d,e,f} = (a, d, e, b, c, f);$$

Remarkably, these three cases completely characterize compatible nondegenerate partitions.

Theorem (Nick Early, Lukas Kühne, LM)

Let $\mathbf{S}$ and $\mathbf{T}$ be two nondegenerate ordered set partitions. Then $\mathbf{S}$ and $\mathbf{T}$ are not compatible if and only if they are 4- or 6-interlaced.

In particular, $\mathbf{S}$ and $\mathbf{T}$ are compatible if and only if $\mathbf{S}_I$ and $\mathbf{T}_I$ are compatible for any I of size at most 6.

Cyclic pattern avoidance

A pair of nondegenerate set partitions (or cyclic orders) $\mathbf{S}$ and $\mathbf{T}$ defines a cyclic permutation $\pi_{\mathbf{S},\mathbf{T}}$. Moreover, $\mathbf{S}$ and $\mathbf{T}$ are 4–interlaced if $\pi_{\mathbf{S},\mathbf{T}}$ (cyclically) contains the pattern 1432 and 6-interlaced if it contains the patterns 125634 or 145236. Thus, nondegenerate set partitions $\mathbf{S}$ and $\mathbf{T}$ are compatible if and only if $\pi_{\mathbf{S},\mathbf{T}}$ is avoiding the above three patterns.

The cyclohedron and the assosiahedron

One can show that the cyclohedron C_n and the assosiahedron A_n are Minkowski sums of compatible alcoved simplices and thus are alcoved polytopes:

The cyclohedron is normally equivalent to the Minkowski sum over all coarsenings of the OSP $(1, 2, \ldots, n)$ such that at most one block has more than one element.

The associahedron normally equivalent to the Minkowski sum over all coarsenings of the OSP $(1, 2, \ldots, n)$ such that at most one block has more than one element and n is in this largest block.

In particular we get:

$$C_4 = \Delta_{(1,2,3,4)} + \Delta_{(1,2,34)} + \Delta_{(1,23,4)} + \Delta_{(12,3,4)} + \Delta_{(2,3,41)} + \Delta_{(1,234)} + \Delta_{(123,4)} + \Delta_{(3,412)} + \Delta_{(2,341)}.$$

$$A_4 = \Delta_{(1,2,3,4)} + \Delta_{(1,2,34)} + \Delta_{(2,3,41)} + \Delta_{(1,234)} + \Delta_{(3,412)} + \Delta_{(2,341)}.$$