Plücker inequalities for weakly separated coordinates in TNN Grassmannians

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Abstract

The fundamental connections of the Grassmannian with both weak separability and Plücker relations are well known. On the totally nonnegative (TNN) part of the Grassmannian, we discover the intrinsic connection between weak separability and Plücker relations. In particular, we show that certain natural sums of terms in a long Plücker relation for pairs of weakly separated Plücker coordinates oscillate around 0 over the TNN Grassmannian. This generalizes the classical oscillating inequalities by Gantmacher–Krein (1941) and recent results on TNN matrix inequalities by Fallat–Vishwakarma (2024). In fact we obtain a characterization of weak separability, by showing that no other pairs of Plücker coordinates satisfy this property. In summary, this uncovers connections between weak separability, Plücker relations, and Temperley–Lieb immanants, and provides a natural and general class of additive inequalities in TNN Grassmannians.

Totally nonnegative matrices

Definition 1. A real matrix A is **totally nonnegative (TNN)** if $\det B \geq 0$ for all square submatrices B of A.

The $n \times n$ TNN matrices identify with directed acyclic planar networks with nonnegative weights with source and sink $\{1, \cdots, n\}$. (Gessel–Viennot, *Adv. Math.* 1985; Lindström, *Bull. London Math. Soc.* 1973; Whitney, *J. Anal. Math.* 1952)

Notations 2. Let $1 \leq m \leq n$ be integers.
- $[m, n] := \{m, \dots, n\}$, and $[n] := [1, n]$ whenever $n \geq 1$.
- For an $n \times m$ matrix A, and subsets $P \subseteq [n]$ and $Q \subseteq [m]$, define
  - $A_{P,Q}$ is the submatrix with row indices P and column indices Q.
  - $A_{jk} := A_{[n]\setminus\{j\},[m]\setminus\{k\}}$.
  - $k \pm S = \{k \pm s : s \in S\}$, for all $k \in \mathbb{Z}$.

We begin with Gantmacher–Krein

For a 4×4 TNN matrix A:

$$\begin{aligned}
\det A &= a_{11}\det A_{11} - a_{12}\det A_{12} + a_{13}\det A_{13} - a_{14}\det A_{14}\\
\det A &\leq a_{11}\det A_{11} - a_{12}\det A_{12} + a_{13}\det A_{13}\\
\det A &\geq a_{11}\det A_{11} - a_{12}\det A_{12}\\
\det A &\leq a_{11}\det A_{11}
\end{aligned}$$

More generally, the following holds:

Theorem 3 (Gantmacher–Krein, 1941). *Fix integers $1 \leq l \leq n$. Then for all TNN $A := (a_{ij})_{n\times n}$,*

$$\sum_{k=1}^{l} (-1)^{1+k} a_{1k} \det A_{1k} \begin{cases} \geq \det A & \text{if } l \text{ is odd, and} \\ \leq \det A & \text{otherwise.} \end{cases}$$

These inequalities refine the Laplace expansion along the first row for TNN matrices. Recent work [1] extends this by deriving similar oscillating inequalities from the generalized Laplace expansion along the first d rows.

Theorem 4 ([1]). *Let $1 \leq d \leq n$ be integers, and suppose $Q_{dk} := [n-d, n] \setminus \{n-d+k\}$ for $k \in [0, d]$. Then the following holds for all $l \in [0, d]$ and all $n \times n$ TNN A:*

$$(-1)^{1+l} \sum_{k=0}^{l} (-1)^{1+k} \det A_{[1,d],Q_{dk}} \det A_{[n]\setminus[1,d],[n]\setminus Q_{dk}} \geq 0.$$

In the same work, "Gantmacher–Krein-type" inequalities are obtained for the classical Karlin's identity, thus refining it for TNN matrices:

Theorem 5 ([1]). *Let $n \geq 1$ be an integer. Suppose $T \subseteq [n]$ and $V := \{v_1 < \cdots < v_{n'}\} = [n] \setminus T$. Then for all $A_{n\times n}$ TNN, all $p \in [n]$ with set S such that $S \subseteq [n] \setminus \{p\}$ and $|S| = |T| + 1$, and all $l \in [n']$,*

$$(-1)^{1+l} \sum_{k=1}^{l} (-1)^{1+k} \det A_{S,T\cup\{v_k\}} \det A_{[n]\setminus\{p\},[n]\setminus\{v_k\}} \geq 0.$$

Is there *one* phenomenon behind *all* these inequalities?

Grassmannian and Plücker relations

For integers $1 \leq m \leq n$, $\mathrm{Gr}(m, m+n) :=$ manifold of m dimensional vector subspaces of $\mathbb{R}^{m+n}$. This identifies with the full rank real matrices $A_{(m+n)\times m}$ quotiented on the right by invertible matrices. The Plücker coordinates $\Delta_I(A) := \det A_{I,[m]}$ for ordered $I \in \binom{[m+n]}{m}$ satisfy the well-known Plücker relations:

$$\Delta_{(i_1,\dots,i_m)} \cdot \Delta_{(j_1,\dots,j_m)} = \sum_{k=1}^{m} \Delta_{(j_k,i_2,\dots,i_m)} \cdot \Delta_{(j_1,\dots,j_{k-1},i_1,j_{k+1},\dots,j_m)}$$

for all $i_1, \dots, i_m, j_1, \dots, j_m \in [m+n]$. Here $\Delta_{(i_1,\dots,i_m)} = \Delta_{\{i_1,\dots,i_m\}}$ if $i_1 < \cdots < i_m$ and $\Delta_{(i_1,\dots,i_m)} = (-1)^{\mathrm{sgn}(w)} \Delta_{(i_{w(1)},\dots,i_{w(m)})}$ for all $w \in S_m$.

What you can anticipate

For $k, r \in [1, m]$, $I = (i_1, \dots, i_m)$, $J = (j_1, \dots, j_m)$, $I_{k,r} := (\dots, j_k, \dots)$, $J_{k,r} := (\dots, i_r, \dots)$, where j_k and i_r replace each other respectively:

$$\Delta_I \Delta_J = \sum_{k=1}^{m} \Delta_{I_{k,r}} \Delta_{J_{k,r}} \quad \text{over} \quad \mathrm{Gr}(m, m+n) \quad \text{for all} \quad r \in [1, m].$$

The Plücker inequalities that we discover, **look like**

$$\Delta_I \Delta_J \leq \sum_{k\in M} \Delta_{I_{k,r}} \Delta_{J_{k,r}} \quad \text{over} \quad \mathrm{Gr}^{\geq}(m, m+n) \quad \text{for nice} \quad M \subseteq [1, m].$$

Totally nonnegative Grassmannian

The totally nonnegative Grassmannian $\mathrm{Gr}^{\geq 0}(m, m+n) \subseteq \mathrm{Gr}(m, m+n)$ corresponds to matrices $A_{(m+n)\times m}$ with all $\Delta_I(A) \geq 0$. Here we have:

$$\overline{A} = \begin{pmatrix} A \\ W_0 \end{pmatrix} \in \mathrm{Gr}^{\geq 0}(m, m+n) \quad \text{for all} \quad A_{n\times m} \text{ TNN},$$

where $W_0 := \left((-1)^{i+1} \cdot \delta_{j,m-i+1}\right)_{i,j=1}^{m}$, because $\det A_{P,Q} = \Delta_I(\overline{A})$, where $I := P \cup ([m+1, m+n] \setminus (m+n+1-Q))$. **Moreover** [3],

$$\sum_{I,J} c_{I,J} \Delta_I(\overline{A}) \Delta_J(\overline{A}) \geq 0 \quad \forall A_{n\times m} \quad TNN$$

$$\iff \sum_{I,J} c_{I,J} \Delta_I \Delta_J \geq 0 \quad \text{over} \quad \mathrm{Gr}^{\geq 0}(m, m+n).$$

The main result

Definition 7

Let $1 \leq m \leq n$ be integers and suppose I, J be m element ordered subsets of $[m+n]$. Locate the elements of I, J on the circle with points $1, 2, \dots, m+n$ marked in clockwise order.
- Call I, J *weakly separated* if $I \setminus J$ and $J \setminus I$ can be separated by a chord in the circle.
- Suppose $\eta = |I \setminus J| = |J \setminus I|$. Let $I \setminus J := \{i_1, \dots, i_\eta\}$ and $J \setminus I := \{j_1, \dots, j_\eta\}$, such that $i_1 <_c \cdots <_c i_\eta <_c j_\eta$ and $i_1 <_c j_1 <_c \cdots <_c j_\eta$ in the clockwise order $<_c$ starting from i_1.
- Examples for $m = n = 6$.
  - For $I = (1, 5, 3, 4, 10, 11)$ and $J = (2, 6, 7, 8, 9, 11)$, a cyclical order is $(i_1, i_2, i_3, i_4, i_5) = (10, 1, 3, 4, 5)$ and $(j_1, j_2, j_3, j_4, j_5) = (2, 6, 7, 8, 9)$.
  - And $I = (1, 10, 3, 4, 2, 11)$ and $J = (5, 8, 7, 6, 9, 11)$ are weakly separated.

6	7	$j_2 = 6$	$7 = j_3$
5	8	$i_5 = 5$	$8 = j_4$
4	9	$i_4 = 4$	$9 = j_5$
3	10	$i_3 = 3$	$10 = i_1$
2	$11 \in I \cap J$	$j_1 = 2$	11
1	$12 \notin I \cup J$	$i_2 = 1$	12

Main Theorem A [3]

Consider the following system of oscillating inequalities for $l, r \in [1, \eta]$, over $\mathrm{Gr}^{\geq 0}(m, m+n)$:

$$\mathrm{sgn}(I_{l,r})\,\mathrm{sgn}(J_{l,r}) \sum_{k=1}^{l} \Delta_{I_{k,r}} \Delta_{J_{k,r}} \geq 0 \quad \forall\, l < \eta - r + 1, \text{ and}$$

$$\mathrm{sgn}(I_{l,r})\,\mathrm{sgn}(J_{l,r}) \left(\sum_{k=1}^{l} \Delta_{I_{k,r}} \Delta_{J_{k,r}} - \Delta_I \Delta_J \right) \geq 0 \quad \forall\, l \geq \eta - r + 1.$$

This system holds for all $l, r \in [1, \eta]$ ***iff*** I and J are weakly separated.

Think of these by looking at Plücker relations (for $r_0 = \eta - r + 1$),

$$0 = \sum_{k=1}^{r_0-1} \Delta_{I_{k,r}} \Delta_{J_{k,r}} + \left(\Delta_{I_{r_0,r}} \Delta_{J_{r_0,r}} - \Delta_I \Delta_J\right) + \sum_{k=r_0+1}^{m} \Delta_{I_{k,r}} \Delta_{J_{k,r}}, \quad \text{and}$$

(1) deleting terms from a fixed end; ***(2)*** getting inequalities at each step.

Back to Gantmacher–Krein

New inequalities along the 2nd row:

$$\begin{aligned}
\det A &= -a_{21}\det A_{21} + a_{22}\det A_{22} - a_{23}\det A_{23} + a_{24}\det A_{24}\\
0 &= -a_{21}\det A_{21} + (a_{22}\det A_{22} - \det A) - a_{23}\det A_{23} + a_{24}\det A_{24}\\
0 &\geq -a_{21}\det A_{21} + (a_{22}\det A_{22} - \det A) - a_{23}\det A_{23}\\
0 &\leq -a_{21}\det A_{21} + (a_{22}\det A_{22} - \det A)\\
0 &\geq -a_{21}\det A_{21}
\end{aligned}$$

More generally, (for a special case of a main result not stated here) consider adding the terms appearing in the Laplace expansion along kth row

$$\begin{matrix} (-1)^{k+1} a_{k1} \det A_{k1}, & a_{kk}\det A_{kk}, -\det A, & (-1)^{k+n} a_{kn} A_{kn}, \\ \ddots & \ddots \quad\quad \ddots & \ddots \\ -a_{k,k-1}\det A_{k,k-1}, & -a_{k,k+1}\det A_{k,k+1}, & \end{matrix}$$

one at a time, in **any** given order, such that we obtain inequalities at each step. Then the sequence must (essentially!) be either the forward or the backward of the sequence shown. This is exactly the sequence the cyclical order yields.

Temperley–Lieb immanants

Let $n \geq 2$ be an integer, and define $T_n(2)$ be the $\mathbb{C}$-algebra generated by the "monoid" $\mathcal{B}_n$ formed by $t_1, \dots, t_{n-1}$ subject to relations:

$$\begin{aligned}
t_i^2 &= 2t_i, && \text{for } i \in [1, n-1],\\
t_i t_j t_i &= t_i, && \text{whenever } |i-j| = 1, \text{ and}\\
t_i t_j &= t_j t_i, && \text{whenever } |i-j| \geq 2.
\end{aligned}$$

Using the homomorphism $\sigma : \mathbb{C}[S_n]/(1 + s_1 + s_2 + s_1 s_2 + s_2 s_1 + s_1 s_2 s_3) \to T_n(2)$ via $\sigma : s_i \mapsto t_i - 1$, define the function

$$f_K : S_n \to \mathbb{R} \quad \text{where } f_K(w) \text{ is the coefficient of } K \text{ in } \sigma(w),$$

for all $K \in \mathcal{B}_n$. For the matrix $\mathbf{x} = (x_{ij})_{n\times n}$ of indeterminates, define:

$$\mathrm{Imm}_{f_K}(\mathbf{x}) := \mathrm{Imm}_K(\mathbf{x}) := \sum_{w\in S_n} f_K(w) x_{1,w_1} \dots x_{n,w_n} \in \mathbb{C}[\mathbf{x}].$$

Theorem 8 (Rhoades–Skandera [2]).
- *All Temperley–Lieb immanants $\mathrm{Imm}_K(\mathbf{x})$ are TNN.*
- *For identical multisets $I \uplus J$, the following holds:*

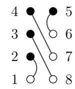
$$\sum_{I,J\in\binom{[2n]}{n}} c_{I,J} \Delta_I(\overline{\mathbf{x}}) \Delta_J(\overline{\mathbf{x}}) \geq 0 \quad \forall\, \mathbf{x}_{n\times n}\ TNN$$

iff *it is a nonnegative linear combination of Temperley–Lieb immanants.*

To keep track of these immanants: the Kauffman diagrams corresponding to each $\Delta_I(\overline{\mathbf{x}})\Delta_J(\overline{\mathbf{x}})$ can be drawn. Consider an example for $n = 4$:

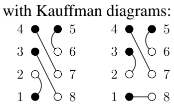
$$\Delta_{\{2,3,4,5\}}(\overline{\mathbf{x}}) \Delta_{\{1,6,7,8\}}(\overline{\mathbf{x}}) = \mathrm{Imm}_{K_1}(\mathbf{x}),$$

where K_1 identifies with the Kauffman diagram:



$$\Delta_{\{1,3,4,5\}}(\overline{\mathbf{x}}) \Delta_{\{2,6,7,8\}}(\overline{\mathbf{x}}) = \mathrm{Imm}_{K_1}(\mathbf{x}) + \mathrm{Imm}_{K_2}(\mathbf{x}),$$

where K_1, K_2 identify with Kauffman diagrams:



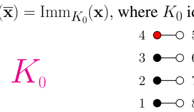
Moreover, there is an inequality:

$$\Delta_{\{1,3,4,5\}}(\overline{\mathbf{x}}) - \Delta_{\{2,6,7,8\}}(\overline{\mathbf{x}}) - \Delta_{\{2,3,4,5\}}(\overline{\mathbf{x}}) \Delta_{\{1,6,7,8\}}(\overline{\mathbf{x}})$$
$$= \mathrm{Imm}_{K_1}(\mathbf{x}) + \mathrm{Imm}_{K_2}(\mathbf{x}) - \mathrm{Imm}_{K_1}(\mathbf{x}) = \mathrm{Imm}_{K_2}(\mathbf{x}) \geq 0$$

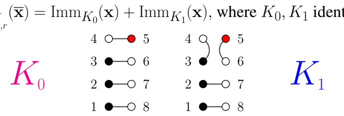
A main proof idea

(Weak separability $\Longrightarrow$ inequalities). $I = (1, 2, 3, 4)$, $J = (5, 6, 7, 8)$ for $n = 4$. Suppose $r = 4$, i.e. $i_4 = 4$. Then $I_{1,r} = (1, 2, 3, 5)$, $J_{1,r} = (4, 6, 7, 8)$; $I_{2,r} = (1, 2, 3, 6)$, $J_{2,r} = (5, 4, 7, 8)$; $I_{3,r} = (1, 2, 3, 7)$, $J_{3,r} = (5, 6, 4, 8)$; $I_{4,r} = (1, 2, 3, 8)$, $J_{4,r} = (5, 6, 7, 4)$.

$\Delta_{I^\uparrow}(\overline{\mathbf{x}}) \Delta_{J^\uparrow}(\overline{\mathbf{x}}) = \mathrm{Imm}_{K_0}(\mathbf{x})$, where K_0 identifies with:

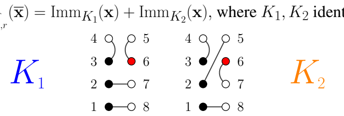


$\Delta_{I^\uparrow_{1,r}}(\overline{\mathbf{x}}) \Delta_{J^\uparrow_{1,r}}(\overline{\mathbf{x}}) = \mathrm{Imm}_{K_0}(\mathbf{x}) + \mathrm{Imm}_{K_1}(\mathbf{x})$, where K_0, K_1 identify with:



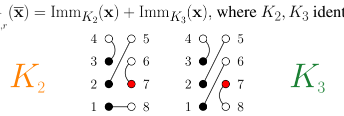
$$\Delta_{I^\uparrow_{1,r}}(\overline{\mathbf{x}}) \Delta_{J^\uparrow_{1,r}}(\overline{\mathbf{x}}) - \Delta_{I^\uparrow}(\overline{\mathbf{x}}) \Delta_{J^\uparrow}(\overline{\mathbf{x}})$$
$$= \mathrm{Imm}_{K_0}(\mathbf{x}) + \mathrm{Imm}_{K_1}(\mathbf{x}) - \mathrm{Imm}_{K_0}(\mathbf{x}) \geq 0.$$

$\Delta_{I^\uparrow_{2,r}}(\overline{\mathbf{x}}) \Delta_{J^\uparrow_{2,r}}(\overline{\mathbf{x}}) = \mathrm{Imm}_{K_1}(\mathbf{x}) + \mathrm{Imm}_{K_2}(\mathbf{x})$, where K_1, K_2 identify with:



$$\left(\Delta_{I^\uparrow_{1,r}}(\overline{\mathbf{x}}) \Delta_{J^\uparrow_{1,r}}(\overline{\mathbf{x}}) - \Delta_{I^\uparrow}(\overline{\mathbf{x}}) \Delta_{J^\uparrow}(\overline{\mathbf{x}})\right) - \Delta_{I^\uparrow_{2,r}}(\overline{\mathbf{x}}) \Delta_{J^\uparrow_{2,r}}(\overline{\mathbf{x}})$$
$$= \mathrm{Imm}_{K_1}(\mathbf{x}) - \left(\mathrm{Imm}_{K_1}(\mathbf{x}) + \mathrm{Imm}_{K_2}(\mathbf{x})\right) \leq 0.$$

$\Delta_{I^\uparrow_{3,r}}(\overline{\mathbf{x}}) \Delta_{J^\uparrow_{3,r}}(\overline{\mathbf{x}}) = \mathrm{Imm}_{K_2}(\mathbf{x}) + \mathrm{Imm}_{K_3}(\mathbf{x})$, where K_2, K_3 identify with:



$$\left(\Delta_{I^\uparrow_{1,r}}(\overline{\mathbf{x}}) \Delta_{J^\uparrow_{1,r}}(\overline{\mathbf{x}}) - \Delta_{I^\uparrow}(\overline{\mathbf{x}}) \Delta_{J^\uparrow}(\overline{\mathbf{x}})\right) - \Delta_{I^\uparrow_{2,r}}(\overline{\mathbf{x}}) \Delta_{J^\uparrow_{2,r}}(\overline{\mathbf{x}}) + \Delta_{I^\uparrow_{3,r}}(\overline{\mathbf{x}}) \Delta_{J^\uparrow_{3,r}}(\overline{\mathbf{x}})$$
$$= -\mathrm{Imm}_{K_2}(\mathbf{x}) + \left(\mathrm{Imm}_{K_2}(\mathbf{x}) + \mathrm{Imm}_{K_3}(\mathbf{x})\right) \geq 0.$$

The last inequality is the Plücker relation itself. □

Link to [3]

References

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[2] Brendon Rhoades and Mark Skandera. Temperley–Lieb immanants. *Ann. Comb.*, **9**(4) (2005), pp. 451–494.

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