The restriction problem and the Frobenius transform

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Restriction coefficients

Let $n \geq 0$ and let λ be a partition with at most n parts. There is a corresponding irreducible $GL_n(\mathbb{C})$-module: the Schur module $\mathbb{S}^\lambda\mathbb{C}^n$. Because the symmetric group $\mathfrak{S}_n$ embeds in $GL_n(\mathbb{C})$ by permutation matrices, one may ask: how does the restriction of $\mathbb{S}^\lambda\mathbb{C}^n$ to $\mathfrak{S}_n$ decompose into irreducible $\mathfrak{S}_n$-modules?

In other words, let λ and μ be partitions and let $n = |\mu|$. What is the value of the *restriction coefficient*

$$r_\lambda^\mu = \dim \operatorname{Hom}_{\mathfrak{S}_n}(V_\mu, \mathbb{S}^\lambda\mathbb{C}^n),$$

where V_μ is the Specht module corresponding to the partition μ?

While there are many known formulas for the restriction coefficient r_λ^μ, no combinatorial interpretation is known. The problem of finding such a combinatorial interpretation is known as the *restriction problem*.

Here is a sampling of recent results about the restriction coefficients r_λ^μ.

- In 2021, Heaton, Sriwongsa, and Willenbring proved the following nonvanishing result: for all positive integers $m, n > 1$ and all $\mu \vdash n$, there exists a two-row partition $\lambda = (\lambda_1, \lambda_2) \vdash mn$ such that $\lambda_1 - \lambda_2 \leq m$ and $r_\lambda^\mu > 0$ [1].
- In 2021, Orellana and Zabrocki introduced the *irreducible character basis* $\{\tilde{s}_\lambda\}_\lambda$ of the ring of symmetric functions and used it to provide an algorithm for computing r_λ^μ [7].
- In 2024, Narayanan, Paul, Prasad, and Srivastava found a combinatorial interpretation for r_λ^μ in the case that μ has one column and λ is either a hook shape or has at most two columns [5].

Notation

We will use the following terminology and notation. Definitions can be found in any standard reference on the theory of symmetric functions.

- The *ring of symmetric functions* Λ and the *ring of symmetric power series* $\overline{\Lambda}$.
- The *monomial*, *elementary*, *homogeneous*, *power sum*, and *Schur* symmetric functions, denoted $m_\lambda, e_\lambda, h_\lambda, p_\lambda, s_\lambda$ respectively.
- The *Hall inner product* $\langle \cdot, \cdot \rangle \colon \Lambda \times \overline{\Lambda} \to \mathbb{Z}$.
- The *plethysm* $f[g]$, where $f, g \in \overline{\Lambda}$.
- The *Littlewood–Richardson coefficient* $c_{\lambda\mu}^\nu$ and *Littlewood–Richardson tableaux*.
- The *Kronecker product* $* \colon \overline{\Lambda} \times \overline{\Lambda} \to \overline{\Lambda}$.

Vanishing of restriction coefficients

When do we have $r_\lambda^\mu = 0$? This question is partially answered by the following theorems.

For partitions λ, μ, let $\lambda \cap \mu$ denote the partition whose Young diagram is the intersection of the Young diagrams of μ and λ. Explicitly, $\ell(\lambda \cap \mu) = \min(\ell(\lambda), \ell(\mu))$ and $(\lambda \cap \mu)_i = \min(\lambda_i, \mu_i)$ for all i.

Theorem ([3, Theorem 1.2])

Let λ, μ be partitions. If the restriction coefficient r_λ^μ does not vanish, then $|\lambda \cap \hat{\mu}| \geq 2|\hat{\mu}| - |\lambda|$, where $\hat{\mu} = (\mu_2, \ldots, \mu_{\ell(\mu)})$ is the partition formed by removing the first part of μ.

For a partition μ, let $D(\mu)$ denote the size of the Durfee square of μ. That is, $D(\mu)$ is the largest integer d such that $\mu_d \geq d$.

Theorem ([3, Theorem 1.5])

Let μ be a partition and let $k \geq 1$ be an integer. The following are equivalent:

1. There exists a partition λ such that $\lambda_1 \leq k$ and $r_\lambda^\mu > 0$.
2. $D(\mu) \leq 2^{k-1}$.

In particular, if λ and μ are partitions with $D(\mu) > 2^{\lambda_1 - 1}$, then $r_\lambda^\mu = 0$.

A special case: At most three columns

We also solve the restriction problem in the case that the partition λ has at most three columns. It would be interesting to see if this answer can be simplified.

Theorem ([4, Corollary 4.1])

Let λ, μ be partitions with $\lambda_1 \leq 3$. Then, r_λ^μ is the number of tuples $(r, \nu, \lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}, T^{(1)}, T^{(2)})$, where

- $r \geq 0$ is an integer;
- $\nu, \lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}$ are partitions;
- $(-\nu_1 + \nu_2 + \nu_3 - r)/2$ is a nonnegative integer;
- $T^{(1)}$ is a Littlewood–Richardson tableau of shape λ/ν^T and content $\lambda^{(1)}$;
- $T^{(2)}$ is a Littlewood–Richardson tableau of shape $\lambda^{(2)}/\lambda^{(1)}$ and content $((\nu_1+\nu_2-\nu_3-r)/2, (\nu_1-\nu_2+\nu_3-r)/2, (-\nu_1+\nu_2+\nu_3-r)/2)$;
- $\lambda^{(3)}/\lambda^{(2)}$ is a vertical strip with r boxes;
- $\mu/\lambda^{(3)}$ is a horizontal strip.

The Frobenius transform

Our results are proved by introducing a linear map $\mathscr{F} \colon \Lambda \to \overline{\Lambda}$ called the *Frobenius transform*. It can be defined in at least six different ways.

1. The matrix entries of $\mathscr{F}$ in the Schur basis are the restriction coefficients. That is,
$$\mathscr{F}\{s_\lambda\} = \sum_\mu r_\lambda^\mu s_\mu.$$
2. $\mathscr{F}$ is the adjoint to plethysm by $H = 1 + h_1 + h_2 + \cdots$ under the Hall inner product. That is,
$$\langle \mathscr{F}\{f\}, g \rangle = \langle f, g[H] \rangle$$
for all $f, g \in \Lambda$.
3. $\mathscr{F}$ is given by the formula
$$\mathscr{F}\{f\} = \sum_\mu f(\Xi_\mu)\frac{p_\mu}{z_\mu},$$
where the sum is over all partitions μ, and Ξ_μ denotes the sequence
$$1, \exp\left(\frac{2\pi i}{\mu_1}\right), \exp\left(\frac{4\pi i}{\mu_1}\right), \ldots, \exp\left(\frac{2(\mu_1 - 1)\pi i}{\mu_1}\right),$$
$$\vdots$$
$$1, \exp\left(\frac{2\pi i}{\mu_\ell}\right), \exp\left(\frac{4\pi i}{\mu_\ell}\right), \ldots, \exp\left(\frac{2(\mu_\ell - 1)\pi i}{\mu_\ell}\right).$$
4. $\mathscr{F}$ is the unique linear map satisfying $\mathscr{F}\{e_r\} = e_r \cdot H$ for all r, as well as $\mathscr{F}\{fg\} = \mathscr{F}\{f\} * \mathscr{F}\{g\}$ for all $f, g \in \Lambda$.
5. Using the Hall inner product, linear maps $\Lambda \to \overline{\Lambda}$ correspond to elements of $\overline{\Lambda \otimes \Lambda}$. The element of $\overline{\Lambda \otimes \Lambda}$ corresponding to $\mathscr{F}$ is the Frobenius characteristic of the category of finite sets and bijections, considered as a bimodule over the category of finite sets and functions.
6. $\mathscr{F}$ is the decategorification of Joyal's *analytic functor* construction [2].

The surjective Frobenius transform

Let $f \in \Lambda$. Even though $\mathscr{F}\{f\} \in \overline{\Lambda}$ can have infinitely many nonzero coefficients, it only carries a finite amount of information. Recall that $H = 1 + h_1 + h_2 + \cdots$.

Proposition ([3, Proposition 3.15])

Let $f \in \Lambda$. There exists a symmetric function $\mathscr{F}_{\mathrm{Sur}}\{f\} \in \Lambda$ such that $\mathscr{F}\{f\} = \mathscr{F}_{\mathrm{Sur}}\{f\} \cdot H$. Moreover, $\mathscr{F}_{\mathrm{Sur}}\{f\}$ has the same degree and leading term as f.

Computing the Frobenius transform

There is no known combinatorial (subtraction-free) formula that writes $\mathscr{F}\{s_\lambda\}$ as a linear combination of Schur functions. Such a formula would solve the restriction problem. However, there are combinatorial formulas for $\mathscr{F}\{h_\lambda\}$, $\mathscr{F}\{e_\lambda\}$, and $\mathscr{F}\{p_\lambda\}$ [6, Equation (6)] [3, Theorem 1.4]. Here is the formula for $\mathscr{F}\{h_\lambda\}$.

Theorem ([6, Equation (6)], [3, Theorem 1.4(a)])

Let λ be a partition and let $\ell = \ell(\lambda)$ be its length. Then,

$$\mathscr{F}\{h_\lambda\} = \sum_M \prod_{j \in \mathbb{N}^\ell} h_{M(j)},$$

where the sum is over all functions $M \colon \mathbb{N}^\ell \to \mathbb{N}$ such that $\sum_{j \in \mathbb{N}^\ell} j_i M(j) = \lambda_i$ for $i = 1, \ldots, \ell$.

For example,

$$\mathscr{F}\{h_{2,2}\} = (h_1 + h_2 + 3h_{1,1} + 2h_{2,1} + h_{1,1,1} + h_{2,2}) \cdot H.$$

References

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