A new shifted Littlewood–Richardson rule and related developments

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Abstract

As Littlewood–Richardson rules compute linear representation theory of symmetric groups and cohomology of ordinary Grassmannians, shifted Littlewood-Richardson rules compute analogous projective representation theory of symmetric groups and cohomology of orthogonal Grassmannians. Serrano (2010) found a plactic structure for the shifted context that allows for the description of shifted Littlewood–Richardson coefficients in terms of an insertion algorithm by Haiman (1989). We obtain the first rectification algorithm for shifted tableaux agreeing with Haiman's mixed insertion, with the caveat that a rectification order of slides must be respected to yield this insertion. Building on related ideas, we also propose a solution to an open problem by Cho (2013) asking for a satisfactory definition of plactic skew Schur P-functions. This work is joint with Oliver Pechenik.

Shifted Plactic Monoid

The **plactic monoid** is a monoid defined on semistandard Young tableaux which affords the Littlewood–Richardson rule. To be specific, the fact that **Yamanouchi tableaux** are a plactic class closed under right factors organically yields a combinatorial description of the multiplication of Schubert classes of $H^\star(\mathrm{Gr}_k(\mathbb{C}^n))$.

We proved that Serrano's **shifted plactic monoid** [5] is the natural analogue for the multiplication of Schubert classes of $H^\star(\mathrm{OG}(n, 2n+1))$, in the following sense:

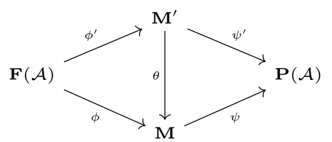
Let $\mathbf{F}(\mathcal{A})$ and $\mathbf{P}(\mathcal{A})$ be the free monoid and plactic monoid on the alphabet $\mathcal{A}$, respectively.

Consider the category $\underline{\mathrm{SPlac}}(\mathcal{A})$ of monoids $\mathbf{M}$ equipped with maps $\phi : \mathbf{F}(\mathcal{A}) \twoheadrightarrow \mathbf{M}$ and $\psi : \mathbf{M} \to \mathbf{P}(\mathcal{A})$ such that

1. $\psi \circ \phi = \kappa$, where κ is projection to the plactic monoid;
2. $H^\star(\mathrm{OG}(n, 2n+1))$ is a commutative subalgebra of $\mathbb{Z}\mathbf{F}(\mathcal{A})$ with generators satisfying weak conditions.
3. Endomorphisms $\omega : \mathbf{F}(\mathcal{A}) \to \mathbf{F}(\mathcal{A})$ that respect the order of the alphabet $\mathcal{A}$ also preserve the fibers of ϕ.
4. for any interval $I \subseteq \mathcal{A}$ and any $w_1, w_2 \in \mathbf{F}(\mathcal{A})$, if $\phi(w_1) = \phi(w_2)$, then

$$\kappa(w_1|_I) = \kappa(w_2|_I).$$

The defining properties of $\underline{\mathrm{SPlac}}(\mathcal{A})$ are mostly analogous to those for a category of plactic monoids essentially present in Schützenberger [4]. The main surprise is the appearance of κ, where one might naturally expect ϕ by analogy. A morphism $\theta : (\mathbf{M}', \phi', \psi') \to (\mathbf{M}, \phi, \psi)$ in $\underline{\mathrm{SPlac}}(\mathcal{A})$ is a monoid homomorphism such that the diagram



(1)

commutes.

This substantiates the shifted plactic monoid as natural from a categorical perspective for the study of $H^\star(\mathrm{OG}(n, 2n+1))$.

Theorem

The shifted plactic monoid $(\mathbf{S}(\mathcal{A}), \sigma, \pi)$ is the initial object of the category $\underline{\mathrm{SPlac}}(\mathcal{A})$.

Barely Yamanouchi Words

Fix a Yamanouchi tableau Y_λ. It is well known that all the words inserting to Y_λ are Yamanouchi words of content λ.

An alternative way to characterize Yamanouchi words is as the set of words with content λ and whose longest increasing subsequence has λ_1 letters, second longest λ_2 letters and so on.

A **hook word** w is a word $w = w_1 \cdot w_2$ where w_1 is strictly decreasing and w_2 weakly increasing.

Considering the set of barely Yamanouchi words with content λ and whose longest hook subsequence consists of λ_1 letters, second longest λ_2 letters, and so on, we obtain the set of **barely Yamanouchi words**.

Theorem

A word $w \in \mathbb{N}^*_{>0}$ with strict content is barely Yamanouchi, if and only if as we read from right to left either of these two conditions:

- $i(w) = (i+1)(w)$ or
- $i(w) = (i+1)(w) + 1$.

holds, where $i(\bullet)$ and $(i+1)(\bullet)$ denote the number of instances of i and $i+1$, respectively.

Barely Yamanouchi words are hence closed under right factors. We use the structure of the shifted plactic monoid to obtain a new combinatorial rule describing the multiplication in $H^\star(\mathrm{OG}(n, 2n+1))$.

Constructed Tableaux

A **constructed tableau** is a left factor of a barely Yamanouchi tableau.

1	2′	3′	3	4′	5′	8′	9′
	4	5′	5	6′	7′	10′	11′
		6	7′	8′	9′		

Figure 1. A constructed tableau of shape $\lambda = (8, 7, 4)$ constructed from $\mu = (4, 2) < \nu = (11, 9, 5)$.

A set of boxes γ is a **generalized rimhook** if it can be partitioned into a vertical strip ξ/π and a horizontal strip θ/η with $\xi \subseteq \eta$. We write $\gamma = \xi/\pi \circledcirc \theta/\eta$.

Let T be a tableau of generalized rimhook shape $\gamma = \xi/\pi \circledcirc \theta/\eta$. We say that T is a **Serrano–Pieri strip** if ξ/π is filled with unprimed letters that increase from top to bottom, θ/η is filled with primed letters that increase from left to right, and each label in ξ/π is less than every label in θ/η.

We say a Serrano–Pieri strip γ can be **extended** if there is another Serrano–Pieri strip $\gamma' = (\xi'/\pi') \circledcirc (\theta'/\eta')$ such that $\gamma' \supsetneq \gamma$ and the letters in γ' (including the primed ones but ignoring their primes) form an interval $(k, e] \supsetneq (\alpha_j, \beta_j]$.

Example 5: To better illustrate the condition that the sequences of the tableaux should be such that none of them can be extended, further consider the invalid tableau.

1	2′	3′	3	4′	5′	8′	9′	10′	11′
	4	4	5′	6′	7′	9′			
		5	6′	7′	8′				

$\longrightarrow$

1	2′	3′	3	4′	5′	8′	9′	10′	11′
	4	4	5′	6′	7′	9′			
		5	6′	7′	8′				

Constructed tableaux can be constructed by laying down successive Serrano–Pieri strips in such a manner that none of the strips can be extended.

Theorem

Let λ, μ, and ν be strict partitions with $|\lambda| + |\mu| = |\nu|$ and $\mu < \nu$. Then, $b^\nu_{\lambda,\mu}$ equals the number of shifted tableaux of shape λ constructed from $\mu < \nu$.

Let $\lambda = (5, 3, 1)$, $\mu = (5, 4)$, and $\nu = (6, 5, 4, 2, 1)$. Then the structure coefficient $b^\nu_{\lambda,\mu}$ equals 2, as witnessed by the 2 constructed tableaux

1	1	1	4′	6
	2	2	5′	
		3		

and

1	1	1	4′	6
	2	2	5	
		3		

(Here, the coloring is redundant, but records steps of the construction process.)

Skew Plactic Schur P-Functions

Cho [1, Open Problem 7.12(1)] asked for a new definition of $\mathcal{P}_{\nu/\mu}$ that makes it a member of $\mathbb{Q}[\mathcal{P}_\lambda]_\lambda$ (in agreement with Serrano's original conjecture); and such that the expansion of $\mathcal{P}_{\nu/\mu}$ in the basis $(\mathcal{P}_\lambda)_\lambda$ can be described in a nice way.

Definition

The *skew plactic Schur P-function* is

$$\mathcal{P}_{\nu/\mu} := \frac{1}{2^{\mathrm{diag}(\nu/\mu)}} \sum_{T \in \mathrm{ShSSYT}(\nu/\mu)'} [F \circ \mathrm{rect}(T)]_{\mathbf{S}} \in \mathbf{S}(\mathcal{A})$$

where $\mathrm{ShSSYT}(\nu/\mu)'$ stands for the set of Q-tableaux of shape λ, F is the map that forgets primes in diagonal positions, $\mathrm{rect}(\bullet)$ signifies Sagan–Worley rectification [7, 3], and $\mathrm{diag}(\nu/\mu)$ denotes the number of diagonal positions in the skew shape ν/μ.

Note that Sagan–Worley's algorithm is employed to rectify T, but after it has been rectified its shifted plactic class (codifying equivalence under Haiman's insertion) is taken.

The following shows that our definition of skew plactic Schur P-functions have the desired properties and are a plactification of the ordinary skew Schur P-functions.

Theorem

Let $\mu < \nu$ be strict partitions. Then $\mathcal{P}_{\nu/\mu} \in \mathbb{Q}[\mathcal{P}_\lambda]_\lambda$ and

$$\mathcal{P}_{\nu/\mu} = \sum_\lambda \frac{2^{\ell(\lambda)}}{2^{\mathrm{diag}(\nu/\mu)}} b^\nu_{\lambda,\mu} \mathcal{P}_\lambda.$$

In particular, the expansion coefficients of $\mathcal{P}_{\nu/\mu}$ in the plactic Schur P-basis are equal to the expansion coefficients of $P_{\nu/\mu}$ in the ordinary Schur P-basis.

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