# Identifying Orbit Lengths for Promotion

Elise Catania [1] Jack Kendrick [2] Heather M Russell [3] Julianna S Tymoczko [4]

[1]University of Minnesota [2]University of Washington [3]University of Richmond [4]Smith College

## Promotion of Standard Young Tableaux

**Promotion** is a function from SYT of fixed shape $\lambda$ to SYT of the same shape. Given $T$, we construct $P(T)$ as follows:

1. Erase 1 in the top left corner of $T$ and leave an empty box.
2. Given the configuration $\begin{array}{|c|c|}\hline & b \\ \hline a & \\ \hline\end{array}$ and $b < a$ then slide $b$ left; else if $a < b$ slide $a$ up.
3. Repeat the above process until there are no nonempty boxes below or to the right of the empty box.
4. Decrement all entries by 1 and insert the largest entry of $T$ into the empty box.

$T =$

| | | | | | |
|---|---|---|---|---|---|
| 1 | 2 | 6 | 7 | 14 | 19 |
| 3 | 8 | 9 | 15 | 18 | 21 |
| 4 | 10 | 11 | 16 | 20 | 23 |
| 5 | 12 | 13 | 17 | 22 | 24 |

$\longrightarrow$

| | | | | | |
|---|---|---|---|---|---|
| 1 | 5 | 6 | 13 | 17 | 18 |
| 2 | 7 | 8 | 14 | 19 | 20 |
| 3 | 9 | 10 | 15 | 21 | 22 |
| 4 | 11 | 12 | 16 | 23 | 24 |

$= P(T)$

Figure 1. The tableau $T$ and its promotion $P(T)$

**Definition.** Suppose $S \subsetneq T$ is a subtableau. We call $S$ *uniformly proper* if:

1. $S$ and $T$ have the same number of rows;
2. $S$ has entries $\{k, k+1, \ldots, k+\ell\}$ for some $k$ and $\ell$.

$S$ a uniformly proper *rectangular* subtableau if in addition all of its rows have the same length. If $T$ contains no uniformly proper subtableau, $T$ is *minimal*. Given a uniformly proper rectangular subtableau $S$ of $T$, we may write $T$ as the horizontal concatenation $T = T_1 S T_2$.

$T$:

| | | | | | |
|---|---|---|---|---|---|
| 1 | 2 | 6 | 7 | 14 | 19 |
| 3 | 8 | 9 | 15 | 18 | 21 |
| 4 | 10 | 11 | 16 | 20 | 23 |
| 5 | 12 | 13 | 17 | 22 | 24 |

$=$ $T_1$:

| | |
|---|---|
| 1 | 2 |
| 3 | |
| 4 | |
| 5 | |

$S$:

| | | |
|---|---|---|
| | 6 | 7 | 14 |
| 8 | 9 | 15 |
| 10 | 11 | 16 |
| 12 | 13 | 17 |

$T_2$:

| | |
|---|---|
| | 19 |
| 18 | 21 |
| 20 | 23 |
| 22 | 24 |

Figure 2. Decomposition of tableau $T$ with uniformly proper subtableau $S$

**Lemma.** Suppose $T$ has a uniformly proper rectangular subtableau $S$ with content $\{k+1, \ldots, k+\ell\}$ for some $k, \ell > 0$. Then, $P(T)$ contains the uniformly proper subtableau $S'$ with content $\{k, k+1, \ldots, k+\ell-1\}$ formed by subtracting 1 from each entry of $S$.

## $m-$Diagrams

An $m-$diagrams is a collection of arcs that satisfy certain noncrossing conditions. There is a simple bijection between SYT and $m-$diagrams [2, 3].

| | | | | | |
|---|---|---|---|---|---|
| 1 | 2 | 6 | 7 | 14 | 19 |
| 3 | 8 | 9 | 15 | 18 | 21 |
| 4 | 10 | 11 | 16 | 20 | 23 |
| 5 | 12 | 13 | 17 | 22 | 24 |

$T$

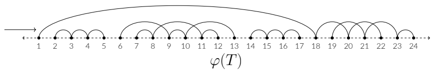



Figure 3. Tableau $T$ and its corresponding $m-$diagram $\varphi(T)$

**Definition.** A *component* is a minimal collection of noncrossing arcs that contains every arc starting and ending at each of its endpoints.

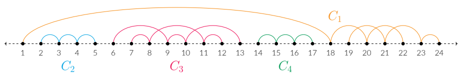



Figure 4. The components of the $m-$diagram $\varphi(T)$.

## Key Observations

1. Minimal uniformly proper rectangular subtableaux of $T$ correspond to *components* of $\varphi(T)$ and so induce a partition of the boundary.
2. Uniformly proper subtableaux are preserved during promotion.
3. The components of $\varphi(P(T))$ are *almost* the components of the left-rotation $\rho(\varphi(T))$, see Figure 5

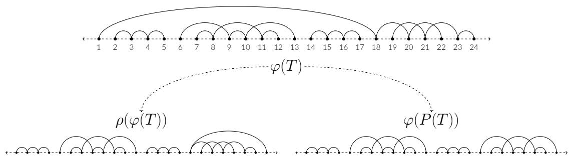



Figure 5. The rotation $\rho(\varphi(T))$ compared to the $m-$diagram $\varphi(P(T))$.

**Idea:** Use rotational symmetry of $\varphi(T)$ to determine the orbit length $|\mathcal{O}(T)|$.

## Algorithm for Computing Orbit Lengths of SYT

For each component $C$ of $\varphi(T)$, construct the corresponding tableau $T_C$. With a slight abuse of language and notation, we define the component promotion $P(T_C)$ of $T_C$ as follows:

1. If $1 \notin \partial C$, form $P(T_C)$ by decrementing each entry of $T_C$ by 1.
2. If $1 \in \partial C$, form $P(T_C)$ by
   1. removing 1 from the top left corner of $T_C$,
   2. performing the sequence of sliding moves for promotion,
   3. decrementing each entry of $T_C$ by 1,
   4. filling the empty box with $|T|$.

**Theorem 1.** The orbit length $|\mathcal{O}(T)|$ can be determined as follows:

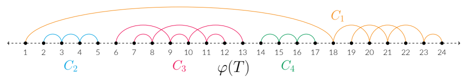
1. Let $N$ be the smallest positive integer such that $\{\rho^N(\partial C_1), \ldots, \rho^N(C_k)\} = \{\partial C_1, \ldots, \partial C_k\}$.
2. Define $\ell$ as

$$\ell = \min\{k \geq 1 : \rho^{kN}(\partial C_i) = \partial C_j \implies P^{kN}(T_{C_i}) = T_{C_j} \forall i\}$$

where $T_{C_i}$ is the tableau corresponding to $C_i$.

Then, the equality $|\mathcal{O}(T)| = \ell N$ holds.

## Example




The following equalities hold:

$$\rho^{12}(\partial C_1) = \partial C_3 \quad \rho^{12}(\partial C_2) = \partial C_4 \quad \rho^{12}(\partial C_3) = \partial C_1 \quad \rho^{12}(\partial C_4) = \partial C_2.$$

So, $N = 12$. As the orbit length $|\mathcal{O}(T)|$ must divide 24, either $|\mathcal{O}(T)| = 12$ or $|\mathcal{O}(T)| = 24$ and so $\ell = 1$ or $\ell = 2$. We calculate the 12-fold promotions of each tableau $T_{C_i}$ :

$T_{C_1}$:

| | |
|---|---|
| 1 | 19 |
| 18 | 21 |
| 20 | 23 |
| 22 | 24 |

$T_{C_2}$:

| |
|---|
| 2 |
| 3 |
| 4 |
| 5 |

$T_{C_3}$:

| | |
|---|---|
| 6 | 7 |
| 8 | 9 |
| 10 | 11 |
| 12 | 13 |

$T_{C_4}$:

| |
|---|
| 14 |
| 15 |
| 16 |
| 17 |

$\xrightarrow{P^{12}}$

$P^{12}(T_{C_1})$:

| | |
|---|---|
| 6 | 7 |
| 8 | 9 |
| 10 | 11 |
| 12 | 13 |

$P^{12}(T_{C_2})$:

| |
|---|
| 14 |
| 15 |
| 16 |
| 17 |

$P^{12}(T_{C_3})$:

| | |
|---|---|
| 1 | 19 |
| 18 | 21 |
| 20 | 23 |
| 22 | 24 |

$P^{12}(T_{C_4})$:

| |
|---|
| 2 |
| 3 |
| 4 |
| 5 |

We see that the equalities hold:

$$P^{12}(\partial C_1) = \partial C_3 \quad P^{12}(\partial C_2) = \partial C_4 \quad P^{12}(\partial C_3) = \partial C_1 \quad P^{12}(\partial C_4) = \partial C_2.$$

It follows that $\ell = 1$ and the orbit length is $|\mathcal{O}(T)| = 12$.

## (Column) Semi-Standard Tableaux

A *(column) semistandard Young tableau (SSYT)* is a filling of the Young diagram for $\lambda$ with the numbers $1, 2, \ldots, n$ so that each number appears at least once, rows strictly increase left-to-right, and columns weakly increase top-to-bottom. If each number $i$ appears $e_i$ times in $T$ then we denote the content $\{1^{e_1}, 2^{e_2}, \ldots, n^{e_n}\}$.

Suppose $T$ has content $\{1^{e_1}, \ldots, n^{e_n}\}$. The map $\psi$ constructs an SYT from $T$ :

1. Relabel each 1 in $T$ sequentially from top to bottom using the set $\{1, 2, \ldots, e_1\}$.
2. For each $i > 1$, relabel each entry $i$ sequentially from top to bottom using the set $\{(e_1 + \ldots + e_{i-1}) + 1, (e_1 + \ldots + e_{i-1}) + 2, \ldots, (e_1 + \ldots + e_{i-1}) + e_i\}$.

$T$:

| | | | | | |
|---|---|---|---|---|---|
| 1 | 2 | 3 | 6 | 9 | 12 |
| 3 | 4 | 5 | 6 | 9 | 12 |
| 3 | 6 | 7 | 8 | 9 | 12 |
| 3 | 6 | 9 | 10 | 11 | 12 |

$\longrightarrow$ $\psi(T)$:

| | | | | | |
|---|---|---|---|---|---|
| 1 | 2 | 3 | 9 | 15 | 21 |
| 4 | 7 | 8 | 10 | 16 | 22 |
| 5 | 11 | 13 | 14 | 17 | 23 |
| 6 | 12 | 18 | 19 | 20 | 24 |

Figure 6. The column SSYT $T$ and its corresponding SYT $\psi(T)$

The promotion $P(T)$ of an SSYT $T$ is formed as follows:

1. Erase 1 from the top left corner and perform sliding moves until there are no nonempty boxes below or to the right.
2. Repeat step 1 a total of $e_1$ times and fill each empty box with $n$.

## Algorithm for Computing Orbit Lengths of SSYT

Given a multiset $\{1^{e_1}, \ldots, r^{e_r}\}$, let $R$ be the smallest integer such that $e_{i+R \mod (r)} = e_i$ for all $1 \leq i \leq r$. We now provide a formula to compute $|\mathcal{O}(T)|$ using $|\mathcal{O}(\psi(T))|$ and $R$.

**Theorem 2.** The orbit length $|\mathcal{O}(T)| = qR$ where $q$ is chosen such that the equality holds:

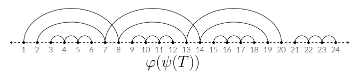
$$q \sum_{i=1}^{R} e_i = \text{lcm}\left(\sum_{i=1}^{R} e_i, |\mathcal{O}(\psi(T))|\right).$$

Since $\psi(T)$ is an SYT, its orbit length $|\mathcal{O}(\psi(T))|$ can be calculated using Theorem 1.

## Example

We now calculate the orbit length $|\mathcal{O}(T)|$ for the tableau $T$ in Figure 6. The content of $T$ is $\{1, 2, 3^4, 4, 5, 6^4, 7, 8, 9^4, 10, 11, 12^4\}$. For each $i$, we have $e_{i+4 \mod 12} = e_i$. Thus, $R = 4$.

The orbit length $|\mathcal{O}(\psi(T))|$ can be calculated using the $m-$diagram $\varphi(\psi(T))$.




In the language of Theorem 1, $N = 6$. By considering each of the five components of $\varphi(\psi(T))$, we calculate that $\ell = 1$ and so $|\mathcal{O}(\psi(T))| = 6$.

Note that $\sum_{i=1}^{R} e_i = 6$. Choosing $q$ to satisfy the equality

$$q \sum_{i=1}^{R} e_i = \ell \cdot 6 = \text{lcm}(6, |\mathcal{O}(\psi(T))|) = \text{lcm}(6, 6),$$

we see that $q = 1$ suffices. Thus the orbit length $|\mathcal{O}(T)| = 1 \cdot 3 = 3$.

## References

[1] Elise Catania, Jack Kendrick, Heather M. Russell, and Julianna Tymoczko. Identifying orbit lengths for promotion, 2025. URL https://arxiv.org/abs/2506.22306.

[2] Heather M Russell. An explicit bijection between semistandard tableaux and non-elliptic sl 3 webs. *Journal of Algebraic Combinatorics*, 38:851–862, 2013.

[3] Julianna Tymoczko. A simple bijection between standard $3 \times n$ tableaux and irreducible webs for $\mathfrak{sl}_3$ *Journal of Algebraic Combinatorics*, 35(4):611–632, 2012.