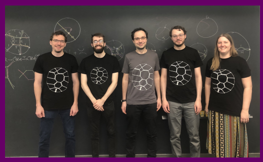


Web Bases for Two-Column Tableaux from Hourglass Plabic Graphs

Christian Gaetz [1] Oliver Pechenik [2] Stephan Pfannerer [2] Jessica Striker [3] Joshua P. Swanson [4]

[1]University of California, Berkeley [2]University of Waterloo [3]North Dakota State University [4]University of Southern California

Motivation

The irreducible representations of the symmetric group S_n are the *Specht modules* S^λ indexed by integer partitions $\lambda \vdash n$. For the case of 3-row rectangles, Kuperberg [7] famously introduced a diagrammatic "web" basis of the Specht module $S^{3\times b}$ (and more generally for other spaces of invariant tensors). We have recently extended this result to 4-row rectagles [4].

Web bases has many important applications to quantum link invariants, cluster algebras, and algebraic geometry. From a combinatorial perspective, a key property of the web basis are that the long cycle $c = (12\dots n)$ acts diagrammatically as a rotation [8].

Based on work by Fraser [2], our new main result [6] is a rotation-invariant web basis for the 2-column rectangular Specht module $S^{r\times 2}$ (and also for more general spaces of tensor invariants).

SL_r-Webs

An **SL_r-web** is a bipartite planar graph embedded in a disc with black boundary vertices, with multi-edges allowed between internal vertices, and whose internal vertices are all r-valent [3].

An SL_r-web is **contracted** if it contains no vertices of simple degree two.

To each SL_r-web W with n boundary vertices, associate a polynomial $[\![W]\!]$ in

$$\mathrm{Inv}((\mathbb{C}^r)^{\otimes n}) = \mathrm{Hom}_{\mathrm{SL}_r(\mathbb{C})}((\mathbb{C}^r)^{\otimes n}, \mathbb{C}) \subset \mathbb{C}[x_{i1}, x_{i2}, \dots, x_{ir} : 1 \le i \le n]$$

which is invariant under the action of SL_r. Recall that as an S_n-module:

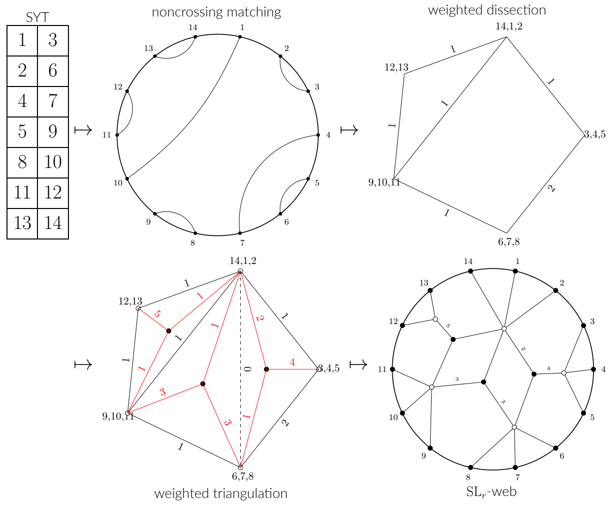
$$\mathrm{Inv}((\mathbb{C}^r)^{\otimes n}) \cong S^{(r\times(n/r))}.$$

The set of invariants associated to all SL_r-webs is spanning the invariant space [1]. A general problem is to pick out a "nice" set of webs giving a basis. Note that $\dim \mathrm{Inv}((\mathbb{C}^r)^{\otimes n}) = |\mathrm{SYT}(r\times(n/r))|$.

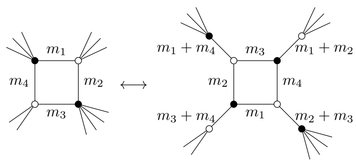
We call n/r the **Plücker degree** of the web.

Fraser's Construction

Fraser [2] gives a map from 2-column rectangular standard Young tableaux with r rows to certain SL_r-webs of Plücker degree two.

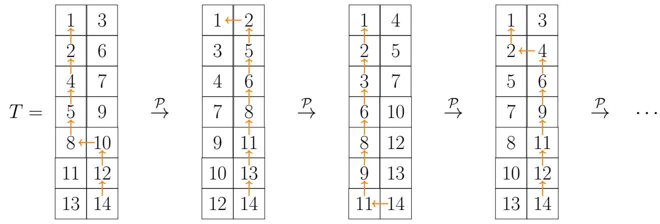


In the construction of the weighted triangulation there are choices involving the "zero"-diagonals. The resulting webs differ by **square moves** and for a fixed tableau all possible webs have the same invariant.



Schützenberger Promotion and Promotion Permutations

Schützenberger **promotion** is defined in terms of jdt-slides.

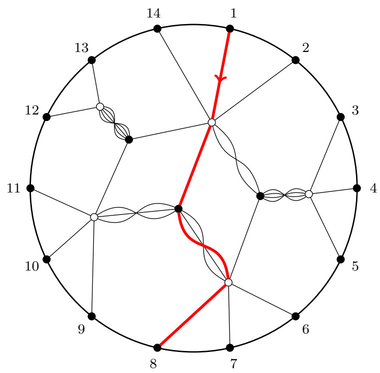


Promotion permutations keep track of the entries in E sliding from row $i+1$ to row i when applying *Schützenberger promotion* [5].

$$\mathrm{prom}_1(T) = 2\ 6\ 4\ 5\ 9\ 7\ 8\ 12\ 10\ 11\ 14\ 13\ 3\ 1 = \mathrm{prom}_6^{-1}(T),$$
$$\mathrm{prom}_2(T) = 4\ 7\ 5\ 9\ 12\ 8\ 10\ 14\ 11\ 13\ 3\ 1\ 6\ 2 = \mathrm{prom}_5^{-1}(T), \text{ and}$$
$$\mathrm{prom}_3(T) = 5\ 9\ 8\ 12\ 14\ 10\ 11\ 1\ 13\ 3\ 6\ 2\ 7\ 4 = \mathrm{prom}_4^{-1}(T).$$

Hourglass Plabic Graphs

An **hourglass plabic graph** is an avatar for an SL_r-web, where each edge of multiplicity m is replaced with m strands twisted like an hourglass.



A key feature of hourglass plabic graphs is that they have $r-1$ **trip permutations** $\mathrm{trip}_1(G), \mathrm{trip}_2(G), \dots \mathrm{trip}_{r-1}(G)$ defined using the rules of the road. For $\mathrm{trip}_i(G)$:

1. Start at boundary vertex b_s,
2. Follow the edges in G and turn at each internal vertex,
3. Take the i-th left at white vertices and i-th right at black vertices.
4. The process ends at boundary vertex b_t. Then set $\mathrm{trip}_i(G)(s) = t$.

Two hourglass plabic graphs G and G' are **equivalent**, $G \sim G'$, if they have the same tuple of trip permutations.

An hourglass plabic graph G is **fully reduced** if its trips avoid certain *double crossings*.

Main Theorems

Theorems: Fraser's construction $\mathcal{F}$ bijectively maps the set of $2\times r$ standard Young tableau to the set of equivalence classes of contracted fully reduced SL_r hourglass plabic graphs of Plücker degree two.

Furthermore, this bijection satisfies $\mathrm{trip}_\bullet(\mathcal{F}(T)) = \mathrm{prom}_\bullet(T)$ and consequently intertwines promotion of tableaux with rotation of hourglass plabic graphs.

Theorem: The invariant polynomials $[\![G]\!]$ of fully reduced SL_r hourglass plabic graphs with $2r$ boundary vertices are a **rotation-invariant web basis** for the invariant space $\mathrm{Inv}((\mathbb{C}^r)^{\otimes 2r})$.

This extends our previous results for fully reduced SL_3- and SL_4-webs. Our earlier results hold for arbitrary Plücker degree (number of columns) but fixed rank (number of rows). The new results hold for an arbitrary rank but fixed Plücker degree.

References

[1] Sabin Cautis, Joel Kamnitzer, and Scott Morrison. Webs and quantum skew Howe duality. *Math. Ann.*, 360(1-2):351–390, 2014.

[2] Chris Fraser. Webs and canonical bases in degree two. *Combin. Theory*, 3(3):Paper No. 11, 26 pages, 2023.

[3] Chris Fraser, Thomas Lam, and Ian Le. From dimers to webs. *Trans. Amer. Math. Soc.*, 371(9):6087–6124, 2019.

[4] Christian Gaetz, Oliver Pechenik, Stephan Pfannerer, Jessica Striker, and Joshua P. Swanson. Rotation-invariant web bases from hourglass plabic graphs. *preprint*, 2023.

[5] Christian Gaetz, Oliver Pechenik, Stephan Pfannerer, Jessica Striker, and Joshua P. Swanson. Promotion permutations for tableaux. *Comb. Theory*, 4(2):Paper No. 15, 56, 2024.

[6] Christian Gaetz, Oliver Pechenik, Stephan Pfannerer, Jessica Striker, and Joshua P Swanson. Web bases in degree two from hourglass plabic graphs. *International Mathematics Research Notices*, 2025(13):rnaf189, 07 2025.

[7] Greg Kuperberg. Spiders for rank 2 Lie algebras. *Comm. Math. Phys.*, 180(1):109–151, 1996.

[8] T. Kyle Petersen, Pavlo Pylyavskyy, and Brendon Rhoades. Promotion and cyclic sieving via webs. *J. Algebraic Combin.*, 30(1):19–41, 2009.