Domino Tilings and Macdonald Polynomials

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Introduction

The study of Macdonald polynomials has produced many interesting combinatorial objects, often expressed as weighted sums over sets of lattice paths. Perhaps the most famous and well-studied such objects are the q,t-Catalan numbers introduced by Garsia and Haiman [4], which can be defined combinatorially as the sum over Dyck paths weighted by the area and dinv statistics. Among the many generalizations of the q,t-Catalan numbers are the extension to Schröder paths defined by Egge, Haglund, Killpatrick, and Kremer [2], and to nested families of Dyck paths due to Loehr and Warrington [6]. All of these objects have natural interpretations in terms of Macdonald polynomials often expressed via the *nabla operator* ∇ on symmetric functions introduced by Bergeron and Garsia [1]. In this work, we study the common generalization to nested families of Schröder paths and their connection to domino tilings.

The Nabla Operator ∇

Let $\mathcal{R}$ be a commutative ring.

- A function (formal power series) $f \in \mathcal{R}[x_1, x_2, \dots]$ is symmetric if
$$f(x_{\sigma(1)}, x_{\sigma(2)}, \dots) = f(x_1, x_2, \dots),$$
for all permutations σ on the positive integers.
- Let $\Lambda_{\mathcal{R}}$ be the collection of symmetric functions and $\Lambda^n_{\mathcal{R}} = \{f \in \Lambda_{\mathcal{R}} | \deg(f) = n\}$.
- Two symmetric functions:
$$h_n(x_1, x_2, \dots) = \sum_{1 \le i_1 \le \dots \le i_n} x_{i_1} \dots x_{i_n}, \quad h_0 := 1. \text{ (complete homogeneous)}$$
$$e_n(x_1, x_2, \dots) = \sum_{1 \le i_1 < \dots < i_n} x_{i_1} \dots x_{i_n}, \quad e_0 := 1. \text{ (elementary)}$$
- Schur functions $\{s_\lambda\}_{\lambda \vdash n}$ form an orthonormal basis of $\Lambda^n_{\mathcal{R}}$, i.e. $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu}$.
- Modified Macdonald polynomials $\tilde{H}_\lambda(\mathbf{x}; q, t)$ are symmetric in $\{x_i\}_{i \ge 1}$ with coefficients in $\mathbb{Z}_{\ge 0}[q,t]$. $\{\tilde{H}_\lambda\}_{\lambda \vdash n}$ forms a basis of $\Lambda^n_{\mathcal{R}}$.
- Take $\mathcal{R} = \mathbb{Q}(q,t)$. The nabla operator is a $\mathbb{Q}(q,t)$-linear map ∇ on $\Lambda_{\mathbb{Q}(q,t)}$ such that
$$\nabla(\tilde{H}_\mu) = q^{n(\mu')} t^{n(\mu)} \tilde{H}_\mu, \text{ for all partitions } \mu,$$
where μ' is the conjugate of μ and $n(\mu) = \sum_i (i-1)\mu_i$.

Central Question: what results are obtained when applying ∇ to elements of some basis of $\Lambda_{\mathbb{Q}(q,t)}$ and expanding the result in terms of another basis?

For example: In the Schur expansion $\nabla(s_\lambda) = \sum_\mu a_{\lambda,\mu} s_\mu$, what does $a_{\lambda,\mu}$ count?

Note that the coefficient $a_{\lambda,\mu} = \langle \nabla(s_\lambda), s_\mu \rangle$.

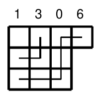
- The q,t-Catalan numbers [Garsia–Haiman, Haglund]
$$\langle \nabla(s_{(1^n)}), s_{(1^n)} \rangle = C_n(q,t) = \sum_{p\text{: Dyck paths}} q^{\mathsf{area}(p)} t^{\mathsf{dinv}(p)}.$$
- The q,t-Schröder numbers [Egge–Haglund–Killpatrick–Kremer, Haglund]
$$\langle \nabla(s_{(1^n)}), h_d e_{n-d} \rangle = S_{n,d}(q,t) = \sum_{p\text{: Schröder paths with } d \text{ diagonal steps}} q^{\mathsf{area}(p)} t^{\mathsf{dinv}(p)}.$$
- [Loehr–Warrington, Blasiak–Haiman–Morse–Pun–Seelinger/Kim–Oh]
$$\langle \nabla(s_\lambda), s_{(1^n)} \rangle = \mathsf{sgn}(\lambda) \sum_{\pi\text{: } \lambda\text{-families of Dyck paths}} q^{\mathsf{area}(\pi)} t^{\mathsf{dinv}(\pi)}$$

Definitions

Let $\lambda \vdash n$ be a partition. A border strip decomposition of λ is defined as follows [Loehr–Warrington]:

(1) decompose the Young diagram of λ by removing successive border strips from the southeastern side.

(2) let n_j denote the number of squares in the border strip ending at the jth box from the right (indexed from $j = 0$) in the top row.



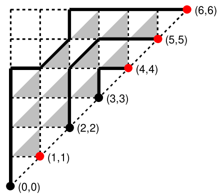
$(n_0, n_1, n_2, n_3) = (6, 0, 3, 1)$.

A λ-families of Schröder paths is a $(k+1)$-tuple $\pi = (\pi_0, \dots, \pi_k)$ where $k = \lambda_1 - 1$, such that

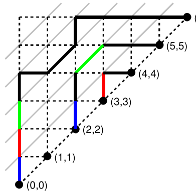
(1) π_i is a lattice path consisting of $(0,1)$, $(1,0)$, and $(1,1)$ steps starting at (i,i) and ending at $(i+n_i, i+n_i)$, which lies weakly above $y = x$,

(2) two distinct paths π_i and π_j do not intersect.

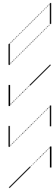
Let $\mathcal{S}_{\lambda,d}$ be the set of λ-families of Schröder paths with d diagonal steps and $\mathcal{S}_\lambda = \bigcup_{d \ge 0} \mathcal{S}_{\lambda,d}$.



A family of Schröder paths π in $\mathcal{S}_{4,3,3}$.



Illustrations of dinv pairs in π.



Dinv pairs.

Let $\pi = (\pi_0, \dots, \pi_k) \in \mathcal{S}_\lambda$. Define

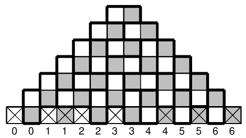
- $\mathsf{area}(\pi_i)$ = the number of shaded triangles in the region bounded by π_i and $y = x$. $\mathsf{area}(\pi) = \sum_{i=0}^k \mathsf{area}(\pi_i)$.
- $\mathsf{adj}(\lambda) = \sum_{j: n_j > 0} (\lambda_1 - 1 - j)$. $\mathsf{sgn}(\lambda) = (-1)^{\mathsf{adj}(\lambda)}$. $\mathsf{dinv}(\pi) = \mathsf{adj}(\lambda) + |\{\text{dinv pairs in } \pi\}|$.

Theorem 1

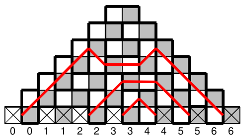
For any partition $\lambda \vdash n$ and $0 \le d \le n$, $\quad \mathsf{sgn}(\lambda) \langle \nabla(s_\lambda), h_d e_{n-d} \rangle = \sum_{\pi \in \mathcal{S}_{\lambda,d}} q^{\mathsf{area}(\pi)} t^{\mathsf{dinv}(\pi)}$.

Domino tilings of R_λ

The region R_λ is obtained by removing from the region the white bottom boxes labeled i and the black bottom boxes labeled $i + n_i$ for each $i = 0, \dots, k$. A domino tiling of R_λ is a collection of dominoes that cover R_λ without gaps or overlaps. Let $\mathcal{T}(R_\lambda)$ the set of domino tilings of R_λ.

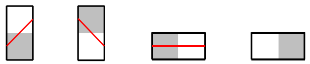


A domino tiling of $R_{(4,3,3)}$.



The corresponding paths under the bijection.

Domino tilings of R_λ are in bijection [7] with λ-families of Schröder paths $\mathcal{S}_\lambda$.



The decoration of four types of dominoes.

Define the generating polynomial of domino tilings of R_λ:
$$P_\lambda(z; q, t) := \sum_{T \in \mathcal{T}(R_\lambda)} z^{\mathsf{diags}(T)} q^{\mathsf{area}(T)} t^{\mathsf{dinv}(T)},$$
where $\mathsf{diags}(T) = |(1,1) \text{ steps of } \pi|$, $\mathsf{area}(T) = \mathsf{area}(\pi)$, and $\mathsf{dinv}(T) = \mathsf{dinv}(\pi)$ under the bijection.

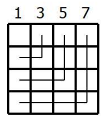
Corollary (the q,t-symmetry of $P_\lambda(z; q, t)$)

For all partitions λ, we have $P_\lambda(z; q, t) = P_\lambda(z; t, q)$.

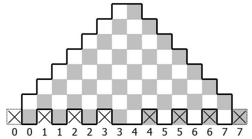
Note: Our proof of Corollary is algebraic based on Theorem 1, finding a bijective proof is open!

Square Shapes $\lambda = (n^n)$

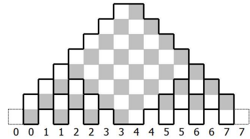
Decomposing $\lambda = (n^n)$ into border strips gives $(n_0, n_1, \dots, n_k) = (2k+1, 2k-1, \dots, 3, 1)$, where $k = n - 1$. After placing forced vertical dominoes, the remaining region reduces to AD_n, the Aztec diamond of order n.



The border strip decomposition of (4^4).

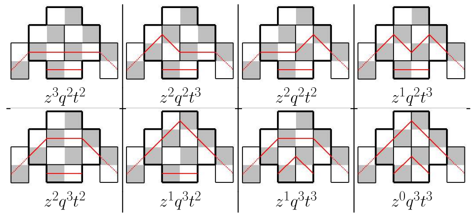


The region $R_{(4,4)}$.



The middle region is AD_4.

Example: $\lambda = (2,2)$. There are 8 domino tilings of AD_2.



Total weight $P_{(2,2)}(z; q, t) = q^2t^2(z+1)(z+q)(z+t)$.

Theorem 2 (A q,t-generalization of the Aztec diamond theorem)

When $\lambda = (n^n)$ is a partition of square shape, then
$$P_{(n^n)}(z; q, t) = (qt)^{n^2(n-1)/2} \prod_{i,j \ge 0 \text{ and } i+j<n} (z + q^i t^j).$$

Discussion:

- When $z = q = t = 1$, this recovers the fact that the number of domino tilings of AD_n is $2^{n(n+1)/2}$.
- Our $\mathsf{diags}(T)$ is the same as $v(T)$ in [3] up to a change of variables.
- Our $\mathsf{area}(T)$ is similar to $rank(T)$ in [3], but they are different.
- Our $\mathsf{dinv}(T)$ appears to be a new statistic.

Proof ingredients for Theorem 1.

- Loehr–Warrington formula.
- Shuffle theorem [5, Chapter 6].

Proof ingredients for Theorem 2.

- Connection between domino tilings of AD_n and alternating sign matrices (ASMs).
- Interpret **area** and **dinv** as new statistics on ASMs.
- Domino shuffling [3].

Key References

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