A generalized LK involution for rectangular tableaux

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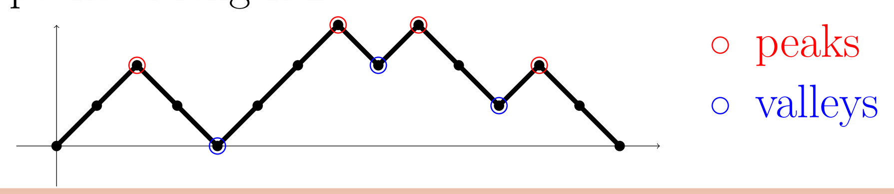
Peaks and valleys in Dyck paths

$\mathcal{D}_n$ = set of Dyck paths of length $2n$



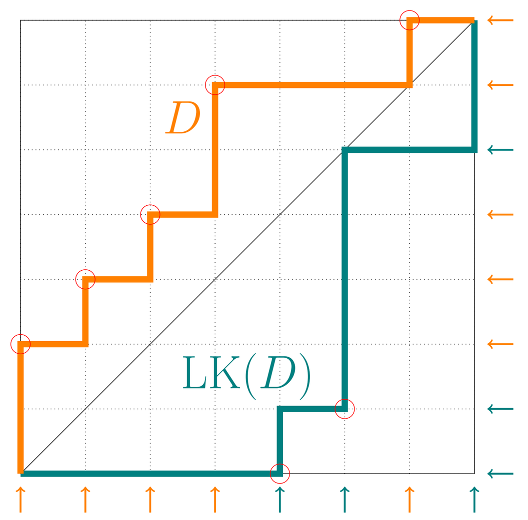
The number of $D \in \mathcal{D}_n$ with h valleys (equivalently, $h+1$ peaks) is the **Narayana number**

$$N(n,h) = \frac{1}{n}\binom{n}{h}\binom{n}{h+1}.$$

We have the symmetry $N(n,h) = N(n,n-h-1)$, i.e.,

$$\#\{D \in \mathcal{D}_n \text{ with } h+1 \text{ peaks}\} = \#\{D \in \mathcal{D}_n \text{ with } n-h \text{ peaks}\}.$$

A bijective proof is given by the **Lalanne–Kreweras (LK) involution**.



Descents and ascents in standard Young tableaux

For $\lambda \vdash N$, let $\mathrm{SYT}(\lambda)$ be the set of standard Young tableaux of shape λ.

i is an ascent if $i+1$ is higher than i

i is a descent if $i+1$ is lower than i

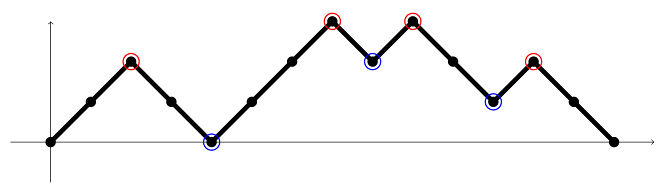
$T =$

1	2	5	7
3	6		
4	8		

$\mathrm{asc}(T) = 2$

$\mathrm{des}(T) = 4$

A simple bijection $\mathcal{D}_n \to \mathrm{SYT}(n,n)$ takes peaks to descents and valleys to ascents:



$\mapsto$

1	2	5	6	7	9	12
3	4	8	10	11	13	14

Let us consider rectangular tableaux with k rows.

The **generalized Narayana numbers** are

$$N(k,n,h) = |\{T \in \mathrm{SYT}(n^k) : \mathrm{asc}(T) = h\}|$$
$$= |\{T \in \mathrm{SYT}(n^k) : \mathrm{des}(T) = h+k-1\}| \quad \text{(Sulanke '04)}$$

Theorem (Sulanke '04)

$$N(k,n,h) = \sum_{\ell=0}^{h} (-1)^{h-\ell}\binom{kn+1}{h-\ell}\prod_{i=0}^{n-1}\prod_{j=0}^{k-1}\frac{i+j+1+\ell}{i+j+1}.$$

Additionally, $N(k,n,h) = N(k,n,(k-1)(n-1)-h)$.

Goal: give a bijective proof of this symmetry.

Arrow encodings of SYT

We encode standard Young tableaux using arrows that describe the placement of the entries:

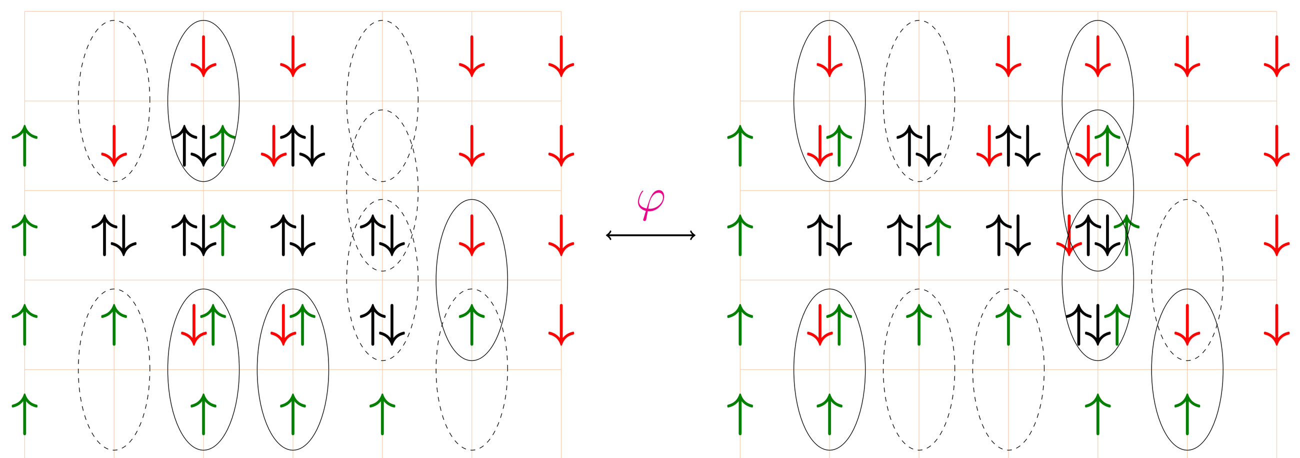
leading $\downarrow$ = leftmost $\downarrow$ in its block

trailing $\uparrow$ = rightmost $\uparrow$ in its block

The bijections

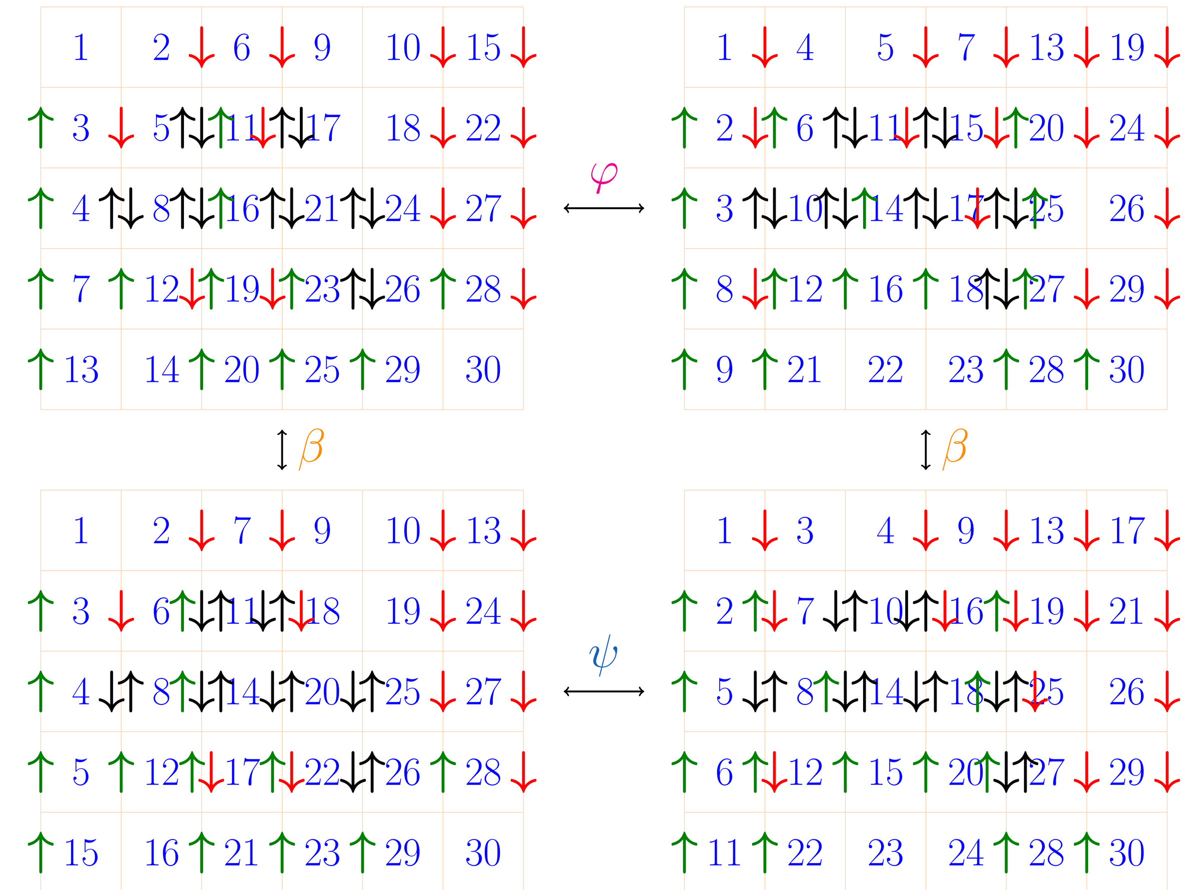
Description of φ: For every pair of arrow blocks $\begin{matrix}A\\B\end{matrix}$ with A immediately above B,

- if A has a leading $\downarrow$ and B has a trailing $\uparrow$, remove these arrows;
- if A has no leading $\downarrow$ and B has no trailing $\uparrow$, add these arrows.



Description of ψ: (Similar to φ, switching the roles of leading and trailing arrows.)

Description of β: Reverse (i.e., read from right to left) each block of arrows.



Main results

Theorem

The maps φ, ψ, β are involutions on $\mathrm{SYT}(n^k)$ satisfying

$$\mathrm{des}(T) + \mathrm{des}(\varphi(T)) = (k-1)(n+1),$$
$$\mathrm{asc}(T) + \mathrm{asc}(\psi(T)) = (k-1)(n-1),$$
$$\mathrm{des}(T) = \mathrm{asc}(\beta(T)) + k - 1, \qquad \mathrm{asc}(T) + k - 1 = \mathrm{des}(\beta(T)).$$

Proposition

The map φ commutes with conjugation, i.e., $\varphi(T)' = \varphi(T')$.

Let $\tau = (n, n-1, \dots, n-k+1)$ be a **truncated staircase**.

Theorem

(A slight variation of) ψ is an involution on $\mathrm{SYT}(\tau)$ satisfying

$$\mathrm{asc}(T) + \mathrm{asc}(\psi(T)) = \frac{(2n-k)(k-1)}{2}.$$

A rowmotion operation on SYT

Rowmotion on Dyck paths (viewed as order ideals of the type A root poset) is a bijection that takes valleys to high peaks (i.e., peaks at height > 1).

We can generalize rowmotion to $\mathrm{SYT}(n^k)$.
Say that i is a high descent of T if $i+1$ is lower than and to the left of i.

Proposition

The map $\rho : \mathrm{SYT}(n^k) \to \mathrm{SYT}(n^k)$ defined by $\rho(T) = \beta(\varphi(\beta(T))')'$ is a bijection satisfying $\mathrm{asc}(T) = \mathrm{hdes}(\rho(T))$.

We get a bijective proof of the symmetry $N(k,n,h) = N(n,k,h)$. More generally...

Theorem

For **any** λ, there is a bijection $\rho : \mathrm{SYT}(\lambda) \to \mathrm{SYT}(\lambda)$ s.t. $\mathrm{asc}(T) = \mathrm{hdes}(\rho(T))$.

Linear extensions of posets

The symmetry $N(k,n,h) = N(k,n,(k-1)(n-1)-h)$ is also a special case of:

Theorem (Stanley '72)

In any graded poset P, the distribution of the number of descents on linear extensions of P (with respect to any given natural labeling) is symmetric.

In '81, Stanley asked for a bijective proof of this symmetry, even in special cases. Our bijection ψ does this for rectangular shapes $P = \mathbf{k} \times \mathbf{n}$ and truncated staircases.

Problem

Give a bijective proof of Stanley's theorem for other graded posets P.