On the reconstruction of trees from their chromatic symmetric function

Michael Gonzalez[1], Rosa Orellana[1], Mario Tomba[1,2]

[1]Dartmouth College [2]The University of Chicago

The Chromatic Symmetric Function

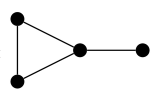
Let G be a graph with vertex set $\{v_1, \dots, v_n\}$.
A *proper coloring* is $\kappa : [n] \to \mathbb{P}$ such that $\kappa(v_i) \neq \kappa(v_j)$ whenever $v_iv_j \in E(G)$
Stanley defined the *chromatic symmetric function* (CSF) as a sum over proper colorings:

$$\mathbf{X}_G = \sum_{\kappa} x_{\kappa(v_1)} \cdots x_{\kappa(v_n)}.$$

The CSF specializes to the chromatic polynomial:

$$\mathbf{X}_G(\underbrace{1, \dots, 1}_{k}, 0, \dots) = \chi_G(k).$$

As an example:

$G =$ 

$$\mathbf{X}_G = 4 \sum_{i \neq j < k} x_i^2 x_j x_k + 24 \sum_{i<j<k<l} x_i x_j x_k x_l$$

The *power-sum* symmetric function of degree r is $p_r \coloneqq \sum_i x_i^r$

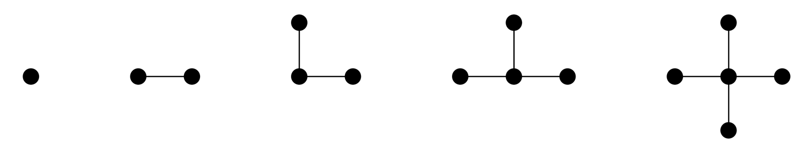
Theorem (Stanley): We have

$$\mathbf{X}_G = \sum_{A \subseteq E} (-1)^{|A|} p_{\lambda(A)},$$

where $\lambda(A)$ has as parts the orders of the connected components of $G|_A$.
Conjecture (Stanley 1995): Let T_1, T_2 be non-isomorphic trees. Then, $\mathbf{X}_{T_1} \neq \mathbf{X}_{T_2}$

The Star Basis

A *star graph* on n vertices $St_{(n)}$ is a tree with $n-1$ leaf vertices and one vertex of degree $n-1$.



St_1 St_2 St_3 St_4 St_5

For a partition $\lambda = (\lambda_1, \dots, \lambda_k)$, we define the *star forest* indexed by λ as follows:

$$St_\lambda = St_{\lambda_1} \cup \cdots \cup St_{\lambda_k}.$$

Denote $\mathfrak{st}_n \coloneqq \mathbf{X}_{St_n}$.
Theorem (Cho-van Willigenburg): In terms of the power-sum symmetric functions:

$$\mathfrak{st}_{n+1} = \sum_{r=0}^{n} (-1)^r \binom{n}{r} p_{(r+1, 1^{n-r})}.$$

Similarly as above, for $\lambda = (\lambda_1, \dots, \lambda_k)$, we have:

$$\mathfrak{st}_\lambda = \mathfrak{st}_{\lambda_1} \cdots \mathfrak{st}_{\lambda_k}$$

Theorem (Cho-van Willigenburg): $\{\mathfrak{st}_\lambda : \lambda \vdash n\}$ is a basis for the algebra of symmetric functions of degree n.
For $\lambda, \mu \vdash n$, we say that $\lambda \leq \mu$ in *lexicographic order* if $\lambda = \mu$ or if

- $\lambda_i = \mu_i$ for $1 \leq i < j$, and
- $\lambda_j < \mu_j$ for some $1 \leq j \leq \ell(\lambda)$

Thus, we write $\mathbf{X}_G$ in increasing lexicographic order:

$$\mathbf{X}_G = \sum_{\lambda \vdash n} c_\lambda \mathfrak{st}_\lambda.$$

We say $e \in E$ is an *internal edge* if both of its endpoints have degree at least two.
Let $I(T)$ denote the set of internal edges of a tree T.
Theorem: In the star-basis expansion, we have:

$$c_{(n-k, 1^k)} = \binom{|I(T)|}{k}$$

Deletion-Near-Contraction

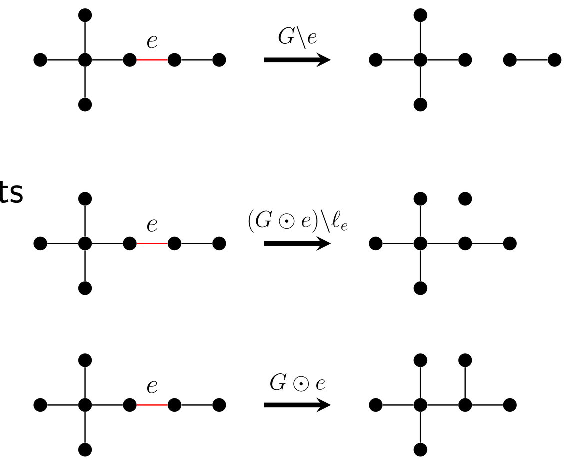
We can write the CSF compactly using the star-basis:

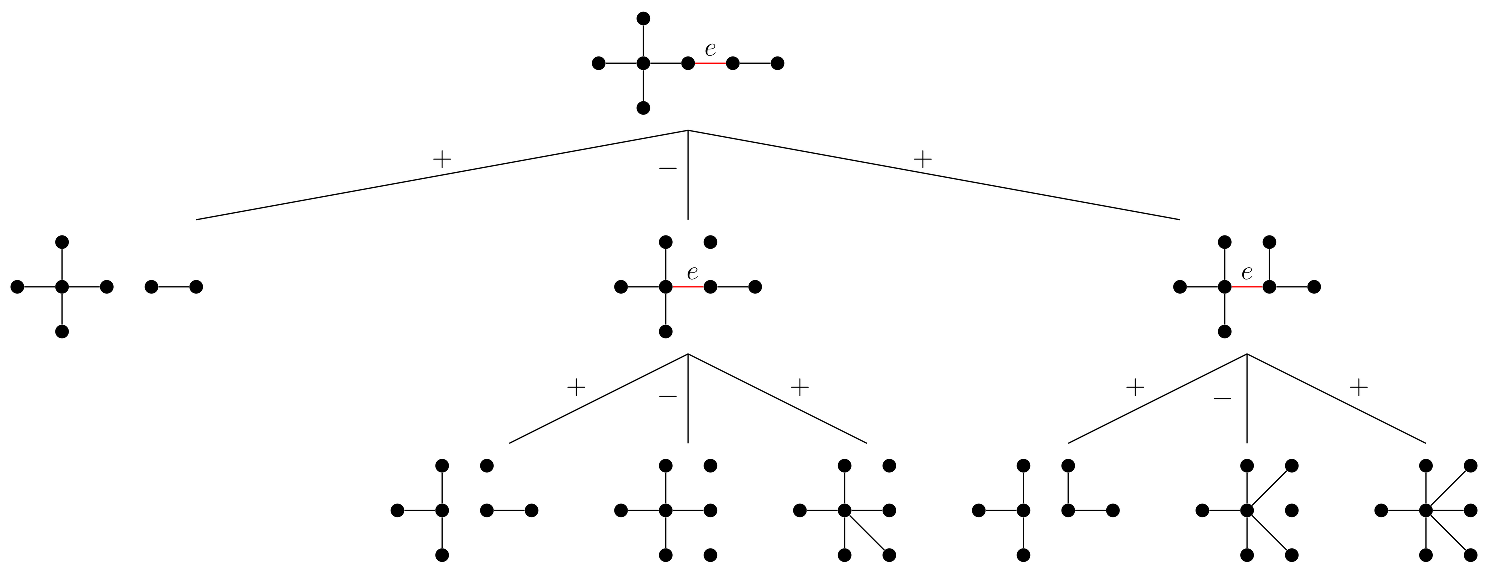
$$\mathbf{X}_G = \sum_{\lambda \vdash n} c_\lambda \, \mathfrak{st}_\lambda$$

The DNC relation expresses the CSF of a forest recursively as a linear combination forests with fewer internal edges [1]:

$$\mathbf{X}_G = \mathbf{X}_{G \setminus e} - \mathbf{X}_{(G \odot e) \setminus \ell_e} + \mathbf{X}_{G \odot e}$$

Recursively applying the DNC relation, we obtain a *DNC tree*.





$$\mathbf{X}_G = -\mathfrak{st}_{(4,2,1)} + \mathfrak{st}_{(4,3)} + \mathfrak{st}_{(5,1,1)} + \mathfrak{st}_{(5,2)} - 2\mathfrak{st}_{(6,1)} + \mathfrak{st}_{(7)}$$

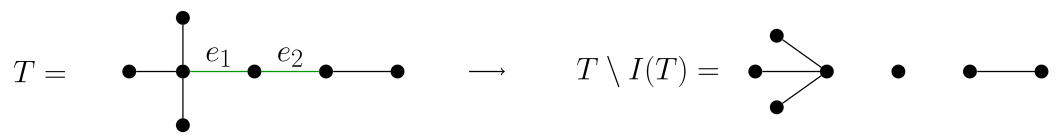
Theorem (Aliste Prieto-de Mier-Orellana-Zamora): We have

$$c_\lambda = (-1)^m |\mathcal{S}_\lambda|,$$

where $\mathcal{S}_\lambda$ is the set of paths in a DNC tree $\mathcal{T}(G)$ that end in a star forest H whose connected components' orders are given by the parts in λ, and m is the number of dot-contractions performed throughout any of these paths.

The Leading Partition

A *leaf component* of a forest F is a connected component of $F \setminus I(F)$.
Let $\lambda_{LC}(F)$ denote the partition whose parts are the orders of the leaf components of F.



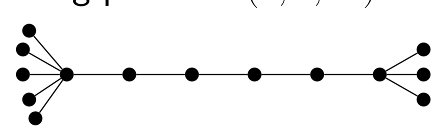
Let $\lambda_{\text{lead}}(\mathbf{X}_F)$ denote the smallest partition in lexicographic order with nonzero coefficient in $\mathbf{X}_F$.
Theorem: For any forest F, we have:

$$\lambda_{\text{lead}}(\mathbf{X}_F) = \lambda_{LC}(F).$$

A *deep vertex* is an internal vertex with no leaves on its neighborhood.
Theorem: Let F be a forest with deep vertices $u_1, \dots, u_m$. Then:

$$c_{\lambda_{\text{lead}}} = (-1)^m \prod_{i=1}^{m} (\deg(u_i) - 1).$$

An *extended bi-star* is a tree consisting of two star graphs whose centers are connected by a path of one or more deep vertices of degree 2.
Example: The following extended bi-star has leading partition $(6, 4, 1^4)$.



Theorem: Let T be a tree of order n. Then, $\lambda_{\text{lead}}(\mathbf{X}_T) = (i, j, 1^{n-i-j})$ for some $i, j > 1$ if and only if T is an extended bi-star with leaf-stars St_i and St_j separated by $n-i-j$ deep vertices of degree 2. Therefore, bi-stars and extended bi-stars are distinguished by their CSF.

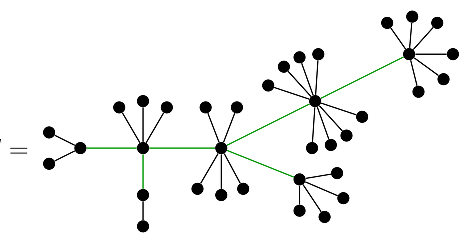
Edge adjacencies

Two leaf components $\mathcal{L}_1, \mathcal{L}_2$ are *adjacent* if their central vertices v_1, v_2 are adjacent.
We say $\mathcal{L}_1, \mathcal{L}_2$ are the *leaf component endpoints* of v_1v_2.
Theorem: Let $\mathbf{X}_T$ be the CSF of a tree T with no deep vertices. Let p, q be two parts in $\lambda_{\text{lead}}(\mathbf{X}_T)$ and let

$$\mu = \text{sort}(p + q, \lambda_{\text{lead}} - \{p, q\}).$$

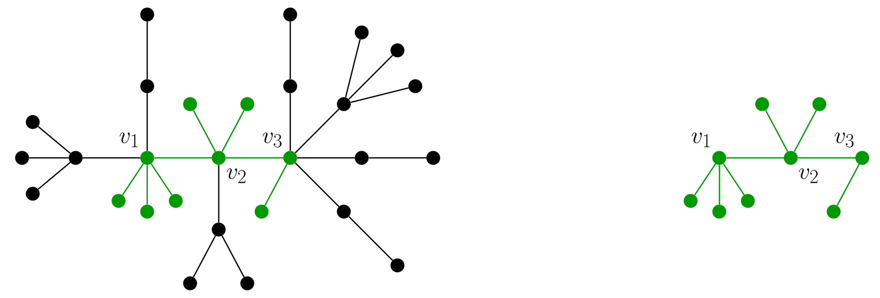
Then, c_μ is the number of edges with leaf component endpoints of orders p and q.
Corollary: Let $\mathbf{X}_T$ be the CSF of a tree T with no deep vertices such that all parts in $\lambda_{\text{lead}}(\mathbf{X}_T)$ are different. Then, T is reconstructible from $\mathbf{X}_T$.
Example Consider $\mathbf{X}_T$ with $\lambda_{\text{lead}}(\mathbf{X}_T) = (9, 7, 6, 5, 4, 3, 2)$.

μ	c_μ	E_μ
$(16, 6, 5, 4, 3, 2)$	1	$\{\{9, 7\}\}$
$(15, 7, 5, 4, 3, 2)$	1	$\{\{9, 6\}\}$
$(11, 9, 7, 4, 3, 2)$	1	$\{\{6, 5\}\}$
$(10, 9, 7, 5, 3, 2)$	1	$\{\{6, 4\}\}$
$(9, 7, 7, 6, 5, 2)$	1	$\{\{4, 3\}\}$
$(9, 7, 6, 6, 5, 3)$	1	$\{\{4, 2\}\}$

$T =$ 

Trees of small diameter

Let $\{v_1, \dots, v_k\}$ be the set of internal vertices of a tree T whose internal degree is strictly greater than one. Let L_i be the set of leaf vertices adjacent to v_i.
The *internal subgraph* of T is the subgraph $\mathcal{I}_T \subseteq T$ induced by $\{v_1, \dots, v_k\} \cup L_1 \cup \cdots \cup L_k$.

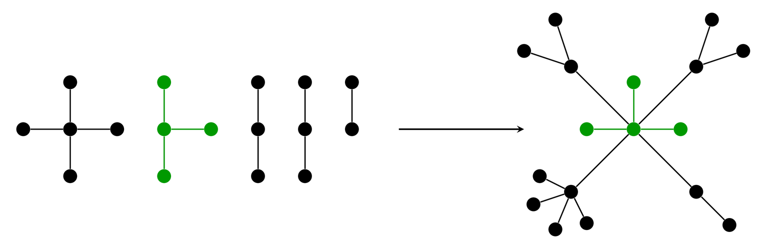


Theorem: Let $\mathbf{X}_T$ be the CSF of a tree with no deep vertices. Let p be a part in $\lambda_{\text{lead}}(\mathbf{X}_T)$. Then, a leaf component of order p is contained in $\mathcal{I}_T$ if and only if:

$$N(p) \coloneqq \sum m_p(E_\mu) \cdot c_\mu > m_p(\lambda_{\text{lead}}),$$

where the sum runs over partitions μ with $c_\mu > 0$ and $\ell(\mu) = \ell(\lambda_{\text{lead}}(\mathbf{X}_T)) - 1$.
Corollary: Trees of diameter 4 can be reconstructed from $\mathbf{X}_T$.
Example: Consider $\mathbf{X}_T$ with $\lambda_{\text{lead}}(\mathbf{X}_T) = (5, 4, 3, 3, 2)$

μ	c_μ	E_μ
$(6, 5, 3, 3)$	1	$\{\{4, 2\}\}$
$(7, 5, 3, 2)$	1	$\{\{4, 3\}\}$
$(9, 3, 3, 2)$	1	$\{\{5, 4\}\}$



Theorem: Let $\mathbf{X}_T$ be the CSF of a tree of diameter 5. Then, T can be reconstructed from $\mathbf{X}_T$.
Proof idea: Let $T \setminus e = T_1 \sqcup T_2$ where $e \in E(\mathcal{I}_T)$. From λ_{lead} and $\mathcal{I}_T$, we can recover $\mathbf{X}_{T \odot e}$ and $\mathbf{X}_{(T \odot e) \setminus \ell_e}$, so we can recover $\mathbf{X}_{T_1}\mathbf{X}_{T_2}$. The smallest partition in this product is, without loss of generality, $\lambda_{\text{lead}}(\mathbf{X}_{T_1})$, and $\lambda_{\text{lead}}(\mathbf{X}_T) - \lambda_{\text{lead}}(\mathbf{X}_{T_1}) = \lambda_{\text{lead}}(\mathbf{X}_{T_2})$. Knowing these partitions and the leaf components in $\mathcal{I}_T$ allow for a reconstruction of T.

References

[1] José Aliste-Prieto, Anna de Mier, Rosa Orellana, and José Zamora, *Marked graphs and the chromatic symmetric function*, SIAM Journal on Discrete Mathematics **37** (2023), no. 3, 1881–1919.

[2] Soojin Cho and Stephanie van Willigenburg, *Chromatic bases for symmetric functions*, Electron. J. Combin. **23** (2016), no. 1, Paper 1.15, 7. MR 3484720

[3] Michael Gonzalez, Rosa Orellana, and Mario Tomba, *The chromatic symmetric function in the star-basis*, 2024.

[4] Richard P. Stanley, *A symmetric function generalization of the chromatic polynomial of a graph*, Adv. Math. **111** (1995), no. 1, 166–194. MR MR1317387 (96b:05174)