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Laboratoire d'algèbre, de combinatoire et d'informatique mathématique

THE h^*-POLYNOMIALS OF TYPE C HYPERSIMPLICES

ANTOINE ABRAM AND JOSE BASTIDAS

UQÀM Université du Québec à Montréal

ROOT SYSTEMS AND ALCOVED POLYTOPES

Let $\Phi = \Phi_{C_n}$ be the root system of type C_n, with simple roots

$$\alpha_1 = 2e_1 \text{ and } , \alpha_i = e_i - e_{i-1}, \text{ for } 2 \le i \le n.$$

For this set of simple roots, the highest root is $\theta = 2e_n$. The **affine Coxeter arrangement** $\mathcal{H}_\Phi$ consists of the hyperplanes

$$H_{\alpha,k} = \{x \in \mathbb{R}^n \mid \langle x, \alpha\rangle = k\}, \text{ for } \alpha \in \Phi^+, k \in \mathbb{Z}.$$

These hyperplanes subdivide $\mathbb{R}^n$ into the **alcoves** of $\mathcal{H}_\Phi$.

Definition (Lam and Postnikov). A polytope $P \subset \mathbb{R}^n$ is an **alcoved polytope** if it is the union of alcoves of $\mathcal{H}_\Phi$. Equivalently, P is defined by inequalities of the form $a_\alpha \le \langle x, \alpha\rangle \le b_\alpha$ for $\alpha \in \Phi^+$ and $a_\alpha, b_\alpha \in \mathbb{Z}$.

Example. The **fundamental alcove**

$$A_\circ := \{x \in \mathbb{R}^n \mid 0 \le \langle x, \alpha\rangle \le 1, \alpha \in \Phi^+\}$$

is an alcoved polytope.

The **fundamental parallelepiped** is

$$\Pi_n := \{x \in \mathbb{R}^n \mid 0 \le 2x_1, x_2 - x_1, \dots, x_n - x_{n-1} \le 1\}.$$

We denote by $\mathcal{I}_{C_n}$ the *weak order* on the alcoves in Π_n with base region $A_\circ$.

The **type C hypersimplices** are

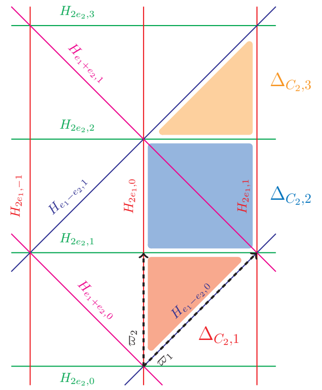
$$\Delta_{C_n,k} := \{x \in \Pi_n \mid k - 1 \le 2x_n \le k\},$$

and the **type C half-open hypersimplices** are

$$\Delta'_{C_n,k} := \{x \in \Pi_n \mid k - 1 < 2x_n \le k\},$$

except at $k = 1$, for which $\Delta'_{C_n,1} := \Delta_{C_n,1}$.

The lattice generated by the vertices of the fundamental alcove $A_\circ = \Delta_{C_n,1}$ is $\Lambda = \frac{1}{2}\mathbb{Z}^n$ and is exactly the collection of vertices of the alcoves of $\mathcal{H}_\Phi$.



TWO FORMULAS FOR THE h^*-POLYNOMIALS OF $\Delta'_{C_n,k}$

Let $\Lambda \subset \mathbb{R}^n$ be a lattice. The **Ehrhart polynomial** of a lattice polytope P is $L_P(r) = L_P^\Lambda(r) := |rP \cap \Lambda|$.

The h^***-polynomial** of P is

$$h^*_P(t) := (1-t)^{\dim(P)+1} \sum_{r \ge 0} L_P(r) t^r.$$

It has positive coefficients (Stanley) and $\mathrm{Vol}(P) = h^*_P(1)$.

Definition 1

For $n \ge 1$, let

$$X_n := \{w \in \mathfrak{B}_n \mid w^{-1}(1) \in [n]\}.$$

Using generating functions, we prove the following.

Theorem 2

For all $n \ge 1$ and $k \ge 1$,

$$h^*_{\Delta'_{C_n,k}}(t) = \sum_{\substack{w \in X_n: \\ \mathrm{fexc}(w) = k-1}} t^{\mathrm{des}_W(w)},$$

where des_W denotes the Coxeter descent of $\mathfrak{B}_n$ and

$$\mathrm{fexc}(w) := 2|\{i \in [n-1] \mid w_i > i\}| + |\{i \in [n] \mid w_i < 0\}|$$

is the **flag-excedance** statistic of Foata and Han.

We obtain another formula of the h^*-polynomials of the hypersimplices using a shelling of their alcove triangulations.

Theorem 3

For $n \ge 1$ and $k \ge 1$,

$$h^*_{\Delta'_{C_n,k}}(t) = \sum_{\substack{w \in X_n: \\ \mathrm{cdes}(w^{-1}) = k}} t^{\mathrm{basc}(w)}.$$

See details below.

ORDER ON (HALF OF) THE SIGNED PERMUTATIONS

For $n \in \mathbb{N}$, let $[\![n]\!] := \{\overline{n}, \cdots, \overline{1}, 1, \cdots, n\}$. For $i, j \in [\![n]\!]$,

$$i \ll j \overset{\text{def}}{\iff} i < k < j \text{ for some } k \in [\![n]\!], \text{ and}$$

i^+ denotes the **cyclic successor** of i in $[\![n]\!]$.

Definition 4

The **big ascent set** of $w \in X_n$ is

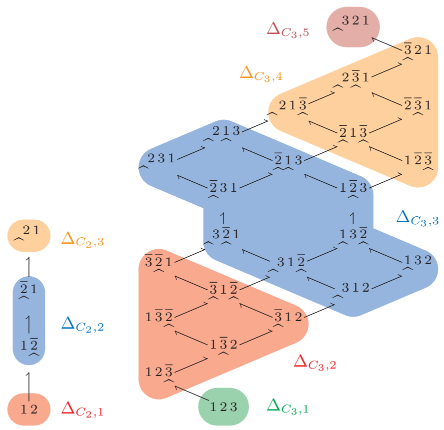
$$\mathrm{BAsc}(w) := \{i \in \{\overline{1}\} \cup [n] \mid w_i \ll w_{i^+}\}.$$

We write $\mathrm{basc}(w) := |\mathrm{BAsc}(w)|$.

Definition 5

Let $\rightharpoonup$ be the following binary relation on X_n: we say $u \rightharpoonup w$, with $w = w_1 w_2 \dots w_{n-1} w_n$, if

$$\begin{cases} \overline{1} \in \mathrm{BAsc}(w) & \text{and } u = \overline{w_1} w_2 \dots w_{n-1} w_n; \text{ or} \\ i \in \mathrm{BAsc}(w) & \text{and } u = \dots w_{i-1} w_{i+1} w_i w_{i+2} \dots; \text{ or} \\ n \in \mathrm{BAsc}(w) & \text{and } u = w_1 w_2 \dots w_{n-1} \overline{w_n}. \end{cases}$$



The posets $(X_n, \le)$ for $n = 2, 3$ with $\frown$ denoting BAsc.

Example. $h^*_{\Delta'_{C_{3,3}}}(t) = 5t + 5t^2$.

Theorem 6

The relation $\rightharpoonup$ is the cover relation of a partial order on X_n, and there is an poset isomorphism $A : X_n \to \mathcal{I}_{C_n}$. Moreover, for every $w \in X_n$, the alcove $A(w)$ lies inside the hypersimplex $\Delta_{C_n,\mathrm{cdes}(w^{-1})}$.

FROM PERMUTATIONS TO ALCOVES

Lam and Postnikov defined the **circular descent statistic**. Explicitly, in our setting, we have the following

$$i \in \mathrm{CDes}(w^{-1}) \iff \begin{array}{l} i^+ \text{ appears before } i \text{ in} \\ w_{\overline{n}} \cdots w_{\overline{1}} w_1 \cdots w_n. \end{array}$$

We prove Theorem 6 by explicitly constructing the vertices $v^1(w), \dots, v^{n+1}(w)$ of $A(w)$ recursively as follows. First, $v^{n+1}(w) = (v_1^{n+1}(w), \cdots, v_n^{n+1}(w))$ where

$$v_i^{n+1}(w) := \left|[\overline{i}, \overline{1}] \cap \mathrm{CDes}(w^{-1})\right|, \quad \text{for } i = 1, 2, \dots, n.$$

Then, for $k \in [n]$, let

$$v^k(w) := v^{k+1}(w) + \tfrac{1}{2} e_{w_k} = v^{n+1}(w) + \tfrac{1}{2}\left(e_{w_k} + \dots + e_{w_n}\right).$$

Example 8. Let $w = \overline{5}\overline{3}\overline{1}\overline{2}\overline{4}4\overline{2}1\overline{3}5$. One has $\mathrm{CDes}(w^{-1}) = \{\overline{3}, \overline{2}, 1, 2\}$. Since $\mathrm{CDes}(w^{-1}) \cap [\overline{5}, \overline{1}] = \{\overline{2}, \overline{3}\}$, then $v^6(w) = (0, 1, 2, 2, 2)$. The other $v^i = v^i(w)$ are

$$v^5 = v^6 + \frac{1}{2} e_5 = (0, 1, 2, 2, \tfrac{5}{2}),$$

$$v^4 = v^5 - \frac{1}{2} e_3 = (0, 1, \tfrac{3}{2}, 2, \tfrac{5}{2}),$$

$$v^3 = v^4 + \frac{1}{2} e_1 = (\tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \tfrac{5}{2}),$$

$$v^2 = v^3 - \frac{1}{2} e_2 = (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{2}, 2, \tfrac{5}{2}) \text{ and,}$$

$$v^1 = v^2 + \frac{1}{2} e_4 = (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{2}, \tfrac{5}{2}, \tfrac{5}{2}).$$

VOLUME AND EULERIAN NUMBERS

Denote the **type BC Eulerian numbers** by

$$B_{n,k} := |\{w \in \mathfrak{B}_n \mid \mathrm{des}_W(w) = k\}|.$$

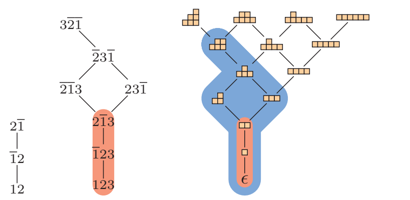
Using a bijection between X_n and its complement and connections between flag statistic and Coxeter descents, we proved the following surprising relation.

Proposition 7. *For all $n \ge 1$ and $k \ge 0$,*

$$B_{n,k} = \mathrm{Vol}(\Delta_{C_n,2k-1}) + 2\,\mathrm{Vol}(\Delta_{C_n,2k}) + \mathrm{Vol}(\Delta_{C_n,2k+1}),$$

where $\mathrm{Vol}(\Delta_{C_n,j}) = 0$ whenever $j < 1$ or $j > 2n - 1$.

LIMITING POSET



Theorem 9. *For all $n \ge 2$, the map $Y_{n-1} \to Y_n : w \to wn$ is a poset embedding. Moreover, the colimit in the category of posets of the diagram $(Y_1 \to Y_2 \to Y_3 \to \dots)$ is the lattice of strict partitions.*

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